

Exact Evolution of the Complete Non-Linear Cosmological Radion

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ABSTRACT: The cosmological two brane model is discussed in detail. An elegant method for deriving the generalised non-linear equations for the interbrane distance is presented. These equations are used to identify equilibrium positions of the radion, and are then linearised in order to determine the nature of these equilibrium solutions. Numerical methods are then employed to determine the complete behaviour of the (unstabilised) cosmological radion for a variety of interesting cases, including radiation and matter dominated non- Z_2 symmetric branes. The implications to theories involving brane collisions are then discussed.

KEYWORDS: Extra Large Dimensions, Cosmology of Theories beyond the SM.

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1. Introduction

Recently there has been considerable interest in the novel suggestion that we live in a Universe that possesses more than four dimensions. The standard model fields are assumed to be confined to a hyper-surface (or 3-brane) embedded in this higher dimensional space, in contrast to the gravitational fields which propagate through the whole of spacetime [1–10]. In order for this to be a phenomenologically relevant model of our universe, standard four-dimensional gravity must be recovered on our brane. There are various ways to do this, the most obvious being to assume that the

extra dimensions transverse to our brane are compact. In this case gravity can be recovered on scales larger than the size of the extra dimensions [5–7]. This is different from earlier proposals since the restrictions on the size of the extra dimensions from particle physics experiments no longer apply, as the standard model fields are confined to the brane. The extra dimensions only have to be smaller than the scale on which gravity experiments have probed, of order 0.1mm at the time of writing. Another way to recover four-dimensional gravity at large distances is to embed a positive tension 3-brane into an AdS_5 bulk [9, 10]. In this scenario four-dimensional gravity is obtained at scales larger than the AdS radius. Randall and Sundrum showed that this could produce sensible gravity even if the extra dimension was not compact.

Several aspects of the above extra dimensional scenarios have since been investigated, and compared with the standard four-dimensional case. The cosmology of a 3-brane in a five-dimensional bulk was studied and its Friedman equation derived and shown to contain several extra terms [11–15]. Perturbations to this homogenous case have been examined [16–19] as have some inflationary models [20, 21], as well as phase transitions, topological defects and baryogenesis [22]. More recently it was shown how to embed the Randall-Sundrum models within supergravities [23–25] and then within string theory compactifications as in [26–29]. The seemingly arbitrary feature of having an AdS bulk spacetime is actually well motivated as it is found as a supersymmetric vacuum to supergravity theories, inspiring several more recent brane world models [30–36]. Generalisations have been made to six dimensions, and it has been demonstrated that both singular and non-singular self-gravitating string defects can localise gravity [37–41].

In this paper we examine the ‘cosmological radion’ which is defined as the inter-brane distance of the cosmologically generalised Randall-Sundrum two brane model. The radion is in general time dependent and in this paper we go on to investigate this time dependent behaviour fully. Many authors impose a stabilisation mechanism on the radion such as [42–46], in order to have a static size of extra dimension. While this is understandable, most of the mechanisms are inserted in an artificial manner and it is therefore interesting to examine the nature of the radion without such purpose built constraints. Several linearised approaches have been made [45, 47–50] however, we will define the radion in a non-perturbative way, and then use the assumptions of homogeneity and isotropy to derive the non-linear equations of motion of the cosmological radion, thus providing a more general and elegant proof to that found in [51].

The radion equations will be used to identify scenarios where stable and unstable interbrane distances exist, and then the equations will be linearised to determine the nature of these equilibrium positions. We then go on to solve numerically for the trajectories of the radion in several cases of interest including when both branes are dS or AdS , when the reference brane possesses a sensible cosmology, when both branes are either radiation or matter dominated, when phase transitions occur, and

when the Z_2 symmetry is broken allowing scenarios with two positive tensions branes. Hence we significantly extend the results found by [51], in which only fairly trivial situations were looked at. A rich variety of behaviour of the radion is found and in order to understand this we present a conceptual argument based on the perspective of a bulk observer, summarised in section 4.

2. Deriving the Complete Non-Linear Equations for the Cosmological Radion

In this section we first describe the generalised Randall-Sundrum two brane scenario before defining the cosmological radion in a non-perturbative manner. The complete non-linear equation for the cosmological radion or interbrane distance $\mathcal{R}(t)$ is then derived for the general case. It is assumed that no Z_2 symmetry exists across either brane and that the bulk possess a non-zero Weyl tensor component and we therefore generalise the derivation presented in [51, 52]. Alternative, mainly linearised approaches can be found in [45, 47–50].

2.1 Cosmological Two Brane Scenario

Since we are interested in cosmological solutions, we take a metric of the form:

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\gamma_{ij}dx^i dx^j + dy^2, \quad (2.1)$$

where y is the coordinate of the fifth dimension and we adopt a brane-based approach where the reference brane is the hyper-surface defined by $y = 0$. Here γ_{ij} is a maximally symmetric 3-dimensional metric with $k = -1, 0, 1$ parameterising the spatial curvature. The metric is found by solving the five-dimensional Einstein's equations which take the form,

$$G_{AB} \equiv R_{AB} - \frac{1}{2}Rg_{AB} = \kappa^2 T_{AB}, \quad (2.2)$$

where R_{AB} is the five-dimensional Ricci tensor, $R = g^{AB}R_{AB}$ the scalar curvature and where we define $\kappa^2 = 1/\widetilde{M}_5^3$, \widetilde{M}_5 being the fundamental (reduced) 5D Planck Mass. The stress-energy-momentum tensor can be written as,

$$T^A{}_B = T^A{}_B|_{brane} + T^A{}_B|_{bulk}. \quad (2.3)$$

Again with cosmology in mind, we assume a homogeneous and isotropic geometry in both branes and this makes it possible to write the first term as:

$$T^A{}_B|_{brane} \equiv \delta(y) \text{diag}(-\rho_0, P_0, P_0, P_0, 0) + \delta(y - \mathcal{R}(t)) \text{diag}(-\rho_2, P_2, P_2, P_2, 0), \quad (2.4)$$

where ρ_0 and P_0 are the energy density and pressure of the reference brane, with similar definitions for the second brane. Here we have defined the position of the

second brane to be at $y = \mathcal{R}(t)$ which is in general time dependent. $\mathcal{R}(t)$ is the inter-brane distance otherwise known as the cosmological radion, and will be investigated throughout this paper. The second term in equation (2.3), which describes the same negative bulk cosmological constant Λ either side of the brane, is of the form,

$$T^A{}_B|_{bulk} = \frac{1}{\kappa^2} \text{diag}(-\Lambda, -\Lambda, -\Lambda, -\Lambda, -\Lambda). \quad (2.5)$$

Substituting the metric given by equation (2.1) into the Einstein equations (2.2) allows one to find the metric coefficients $a(t, y)$ and $n(t, y)$ in the bulk [53], which are used throughout this paper. The Friedmann equation on the reference brane can also be found by using the Einstein equations combined with the Israel junction conditions which relate the jump in the extrinsic curvature tensor to the energy-momentum-tensor on the brane [54–57]. This gives the well known (non- Z_2 symmetric) brane world Friedmann equation [15, 53] (with $a_0 \equiv a(t, y = 0)$),

$$H_0^2 \equiv \left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\Lambda}{6} + \frac{\kappa^4}{36}\rho_0^2 - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4} + \frac{F^2}{\rho_0^2 a_0^8}, \quad (2.6)$$

where \mathcal{C} and F are constants related to the Weyl tensor and the lack of Z_2 symmetry respectively. These constants are in fact related to the black hole masses on either side of the reference brane by $\mathcal{C} = (\mathcal{C}_0 + \mathcal{C}_1)/2$ and $F = 3(\mathcal{C}_1 - \mathcal{C}_0)/4\kappa^2$, where we have defined the black hole mass to the left of the reference brane ($y < 0$) as \mathcal{C}_0 and to the right ($0 < y < \mathcal{R}$) as \mathcal{C}_1 . Below we will also use the mass of the black hole to the right of the second brane ($\mathcal{R} < y$) defined as \mathcal{C}_2 .

In order to avoid the redundancy of several physical constants it is desirable to define a mass scale μ and an energy density scale σ such that,

$$\mu^2 = -\frac{\Lambda}{6}, \quad \sigma = \frac{6\mu}{\kappa^2}. \quad (2.7)$$

We can then deal solely with the dimensionless energy densities (η_0 and η_2) and pressures (p_0 and p_2) of the branes, defined by,

$$\eta_0 = \frac{\rho_0}{\sigma}, \quad \eta_2 = \frac{\rho_2}{\sigma}, \quad (2.8)$$

$$p_0 = \frac{P_0}{\sigma}, \quad p_2 = \frac{P_2}{\sigma}. \quad (2.9)$$

These dimensionless quantities will be used throughout the rest of this paper. We now go on to derive the non-linear equation for $\mathcal{R}(t)$.

2.2 An Elegant Method for Finding the Equation for $\dot{\mathcal{R}}$

Using the dimensionless quantities defined by equations (2.8) and (2.9), the Friedmann equation on the reference brane now looks like,

$$H_0^2(t) = \mu^2 \eta_0^2 - \mu^2 - \frac{k}{a_0^2} + \frac{\mathcal{C}_0 + \mathcal{C}_1}{2a_0^4} + \frac{(\mathcal{C}_1 - \mathcal{C}_0)^2}{16\mu^2 \eta_0^2 a_0^8}, \quad (2.10)$$

where \mathcal{C}_0 and \mathcal{C}_1 are the masses of the Schwarzschild black hole either side of the brane. If the second brane follows a trajectory given by $y = \mathcal{R}(t)$, then the induced four-dimensional metric on the second brane is given by,

$$ds^2 = - \left[n^2(t, \mathcal{R}(t)) - \dot{\mathcal{R}}^2 \right] dt^2 + a^2(t, \mathcal{R}(t)) dx^2, \quad (2.11)$$

$$= -d\tau^2 + a_2^2(\tau) dx^2, \quad (2.12)$$

where the dot represents the derivative with respect to the reference brane time t , the subscript 2 implies evaluation on the second brane at $y = \mathcal{R}$, and τ has been defined as the proper time as seen by an observer on the second brane and is defined by,

$$d\tau = \sqrt{n^2(t, \mathcal{R}(t)) - \dot{\mathcal{R}}^2} dt. \quad (2.13)$$

The expansion rate of the second brane as seen by an observer on our brane is simply,

$$H_2(t) = \frac{1}{a_2} \frac{da_2}{dt} = \left(\frac{\dot{a}}{a} + \frac{a'}{a} \dot{\mathcal{R}} \right)_2, \quad (2.14)$$

where the dash represents the derivative with respect to y , and as seen by an observer on the second brane itself,

$$\mathcal{H}_2(\tau) = \frac{1}{a_2} \frac{da_2}{d\tau} = H_2(t) \frac{dt}{d\tau}, \quad (2.15)$$

$$\Rightarrow \mathcal{H}_2(\tau) = \left(\frac{\dot{a}}{a} + \frac{a'}{a} \dot{\mathcal{R}} \right)_2 \left(n^2 - \dot{\mathcal{R}}^2 \right)_2^{-1/2}. \quad (2.16)$$

This equation for \mathcal{H}_2 is important as it relates the expansion rate of the second brane (with respect to its proper time) to \mathcal{R} and $\dot{\mathcal{R}}$. It does not seem so useful at first, as calculating \mathcal{H}_2 could be difficult. However, it is known that the brane world Friedmann equation given by (2.10) is derived from a purely local analysis and that should we have chosen the second brane to be stationary and at $y = 0$, we would have derived the equivalent Friedmann equation with η_0 replaced by η_2 . This means that $\mathcal{H}_2(\tau)$ must have the following form,

$$\mathcal{H}_2^2(\tau) = \mu^2 \eta_2^2 - \mu^2 - \frac{k}{a_2^2} + \frac{\mathcal{C}_1 + \mathcal{C}_2}{2a_2^4} + \frac{(\mathcal{C}_2 - \mathcal{C}_1)^2}{16\mu^2 \eta_2^2 a_2^8}, \quad (2.17)$$

where we have defined as before $\eta_2 = \rho_2/\sigma$ and \mathcal{C}_1 and \mathcal{C}_2 are again the masses of the black hole to the left and the right of the second brane. It should be understood that equation (2.17) ensures that the second brane evolves according to the junction conditions of the extrinsic curvature tensor, just as equation (2.10) ensures similar behaviour for our brane. At this point we use the following bulk identity which is obtained from the Einstein equations and derived in [12],

$$\left(\frac{\dot{a}}{na} \right)^2 = \frac{a'^2}{a^2} - \mu^2 - \frac{k}{a^2} + \frac{\mathcal{C}_i}{a^4}, \quad (2.18)$$

where as before $i = 0, 1$ or 2 for the regions $y < 0$, $0 < y < \mathcal{R}$, or $y > \mathcal{R}$ respectively. Combining this with equation (2.17) gives, for the $0 < y < \mathcal{R}$ case,

$$\mathcal{H}_2^2 = \left(\frac{\dot{a}}{na} \right)_2^2 - \frac{a_2'^2}{a_2^2} + \left(\mu\eta_2 + \frac{f_2}{\eta_2 a_2^4} \right)^2, \quad (2.19)$$

where the usual notation $f_2 = (\mathcal{C}_2 - \mathcal{C}_1)/4\mu$ has been used. Substituting (2.19) into (2.16) and rearranging gives the following equation for $\dot{\mathcal{R}}$,

$$\frac{a'}{a} + \frac{\dot{a}}{n^2 a} \dot{\mathcal{R}} = \left(\mu\eta_2 + \frac{f_2}{\eta_2 a_2^4} \right) \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{1/2}, \quad (2.20)$$

where we have dropped the subscript ‘2’ as from now on all the metric components are to be evaluated at $y = \mathcal{R}$ unless stated otherwise. This therefore generalises the result derived by [51]. Equation (2.20) can be algebraically solved to give the generalised first order equation for $\dot{\mathcal{R}}$:

$$\dot{\mathcal{R}} = n \left[-\frac{a'\dot{a}}{a^2 n} \pm \mathcal{H}_2 \left(\mu\eta_2 + \frac{f_2}{\eta_2 a_2^4} \right) \right] \left[\frac{a'^2}{a^2} + \mathcal{H}_2^2 \right]^{-1}. \quad (2.21)$$

One has to be careful when considering in which situations solutions exist and in some cases which sign should be chosen in equation (2.21). We need to consider *both* equations (2.20) and (2.16), a fact that was neglected by [51] and led to them having to change the \pm sign in equation (2.21) by hand in order to enable their numerical computation. Equation (2.20) tells us immediately that $-n < \dot{\mathcal{R}} < n$, which just means that the motion of the second brane relative to the first cannot be greater than the local speed of light. With this in mind the same equation now tells us that if a'/a and $\mu\eta_2 + f_2/\eta_2 a_2^4$ are of the same sign, then if $|a'/a| > |\dot{a}/na|$ there will be two solutions, however if $|a'/a| < |\dot{a}/na|$ only one solution exists. On the other hand, if a'/a and $\mu\eta_2 + f_2/\eta_2 a_2^4$ are of the opposite sign, $|a'/a| > |\dot{a}/na|$ implies no solutions and $|a'/a| < |\dot{a}/na|$ implies only one. For the cases where there is one solution only, the overall choice of the \pm in (2.21) should be the sign of $\dot{a}(\mu\eta_2 + f_2/\eta_2 a_2^4)$. Inspecting equation (2.16) we see a similar story: if \mathcal{H}_2 and \dot{a}/na are the same sign, then if $|a'/a|$ is greater/less than $|\dot{a}/na|$ it implies one/two solutions. If \mathcal{H}_2 and \dot{a}/na are of opposite signs then if $|a'/a|$ is greater/less than $|\dot{a}/na|$ it implies one/zero solutions. The overall sign for the single solutions in this case should be that of $\mathcal{H}_2 a'/a$. Therefore these two equations compliment each other: when (2.20) implies there are two solutions, equation (2.16) shows that there is in fact only one (which one depends on the sign of \mathcal{H}_2). This was neglected by [51], and will be used in generating the numerical solutions in section 5.

2.3 Derivation of the $\ddot{\mathcal{R}}$ Equation

In order to investigate equilibrium positions of the second brane it is necessary to derive the equation for $\ddot{\mathcal{R}}$. Taking the time derivative of equation (2.20) and then

adding to it H_2 times equation (2.20) gives,

$$\begin{aligned} \frac{d}{dt} \left(\frac{a'}{a} + \frac{\dot{a}}{an^2} \dot{\mathcal{R}} \right) + \left(\frac{\dot{a}}{a} + \frac{a'}{a} \dot{\mathcal{R}} \right) \left(\frac{a'}{a} + \frac{\dot{a}}{an^2} \dot{\mathcal{R}} \right) = \\ \frac{1}{2} g \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{-1/2} \frac{d}{dt} \left(-\frac{\dot{\mathcal{R}}^2}{n^2} \right) + \frac{dg}{dt} \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{1/2} + g H_2 \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{1/2}, \end{aligned} \quad (2.22)$$

where the function $g(a, \eta_2)$ is given by $g = \mu\eta_2 + f_2/\eta_2 a^4$. Noting first that

$$\frac{dg}{dt} = \mu\dot{\eta}_2 - \frac{f_2}{\eta_2 a^4} \dot{\eta}_2 - 4 \frac{f_2}{\eta_2 a^4} \frac{1}{a} \frac{da}{dt}, \quad (2.23)$$

and the identities: $(1/a)da/dt = H_2$ and $\dot{\eta}_2 = -3H_2(\eta_2 + p_2)$ which is just the energy-momentum conservation equation for the second brane, one then finds that by multiplying (2.22) by $1 - \dot{\mathcal{R}}^2/n^2$ and rearranging, one gets,

$$\begin{aligned} & -\frac{1}{3}G_{05} \left(1 - \frac{\dot{\mathcal{R}}^4}{n^4} \right) - \frac{1}{3}\dot{\mathcal{R}} (G_{55} + G_{00}/n^2) \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right) \\ & + H_2 \left[\frac{\ddot{\mathcal{R}}}{n^2} + \frac{n'}{n} \left(1 - 2\frac{\dot{\mathcal{R}}^2}{n^2} \right) - \frac{\dot{n}}{n} \frac{\dot{\mathcal{R}}}{n^2} \right] \\ = & H_2 \left(-2\mu\eta_2 - 3\mu p_2 + \frac{3f_2 p_2}{\eta_2^2 a^4} \right) \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{3/2}. \end{aligned} \quad (2.24)$$

We will be mainly interested in the case where the bulk possesses a cosmological constant, and therefore from the Einstein equations (2.2) we have $G_{05} = 0$ and $G_{00}/n^2 = -G_{55} = -6\mu^2$. This means that the first two terms on the left hand side of equation (2.24) vanish. In fact a more general argument first proposed by [51] shows that these terms will vanish as long as the bulk energy-momentum tensor satisfies the three-dimensional symmetries of homogeneity and isotropy. From (2.24) we can now see that the generalised equation for $\ddot{\mathcal{R}}$ is given by,

$$\frac{\ddot{\mathcal{R}}}{n^2} + \frac{n'}{n} \left(1 - 2\frac{\dot{\mathcal{R}}^2}{n^2} \right) - \frac{\dot{n}}{n} \frac{\dot{\mathcal{R}}}{n^2} = \left(-2\mu\eta_2 - 3\mu p_2 + \frac{3f_2 p_2}{\eta_2^2 a^4} \right) \left(1 - \frac{\dot{\mathcal{R}}^2}{n^2} \right)^{3/2}. \quad (2.25)$$

Equations (2.21) and (2.25) govern the evolution of \mathcal{R} and will be used to examine the behaviour of the general cosmological radion.

3. The Linearised Radion Equation

In this section the full equations (2.21) and (2.25) that govern the evolution of the cosmological radion are examined analytically. Unfortunately, due to the highly non-linear nature of these equations finding an analytic solution is impossible in all but

the most trivial of cases. Therefore, here the stable solutions will be identified and the nature of their stability discussed along with a few other important features that the solutions must possess. These results will then be compared with the extensive numerical analysis that occurs in the next section.

3.1 Equilibrium Solutions

In order to find the equilibrium solutions, $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are both set to zero in equations (2.21) and (2.25). This restricts the equation of state on the second brane to be of a specific form. It will be shown in general that in order for the size of the fifth dimension to be constant, an equation of state of the form $p_2 = \omega_2 \eta_2$ inevitably leads to a time dependent ω_2 . Setting $\dot{\mathcal{R}} = \ddot{\mathcal{R}} = 0$ in equation (2.21) leads to,

$$\frac{a'}{a} = \mu \bar{\eta}_2 + \frac{f_2}{\bar{\eta}_2 a^4}, \quad (3.1)$$

where a and a' are evaluated at position $y = \mathcal{R}$ and we have defined $\bar{\eta}_2$ to be the dimensionless energy density on the second brane at equilibrium. Rearranging gives,

$$2\mu \bar{\eta}_2 = \frac{a'}{a} \pm \sqrt{\frac{a'^2}{a^2} - \frac{4f_2\mu}{a^4}}. \quad (3.2)$$

This shows that if for example $a'/a < 0$ for all y (as is the case for a cosmologically realistic Z_2 symmetric brane where $a(t, y) = \cosh \mu y - \eta_0 \sinh \mu y$ and $\eta_0 > 1$) and the non- Z_2 symmetry breaking parameter satisfies $0 < f_2 < (aa')^2/4\mu$ then the possible solutions for $\bar{\eta}_2$ are both negative. The case where $f_2 > (aa')^2/4\mu$ of course has no equilibrium solution. If however, $f_2 < 0$ then one of the solutions will be positive, therefore one could have two positive tension branes in equilibrium in a semi-infinite five dimensional space-time. This corresponds to the setup examined in [58] and will be discussed in more detail later. Setting $\dot{\mathcal{R}} = \ddot{\mathcal{R}} = 0$ in equation (2.25) leads to a similar constraint on p_2 ,

$$\frac{n'}{n} = -2\mu \bar{\eta}_2 - 3\mu \bar{p}_2 + \frac{3f_2 \bar{p}_2}{\bar{\eta}_2^2 a^4}, \quad (3.3)$$

where \bar{p}_2 is the dimensionless pressure on the second brane at equilibrium. This constraint can be rewritten using equation (3.1) to give,

$$\bar{p}_2 = \frac{\bar{\eta}_2 \left(\frac{n'}{n} + 2\mu \bar{\eta}_2 \right)}{3 \left(\frac{a'}{a} - 2\mu \bar{\eta}_2 \right)}. \quad (3.4)$$

This demonstrates that except for a few trivial situations, the equilibrium requirement forces ω_2 to be time dependent and therefore for the second brane to have an ‘unnatural’ equation of state. Note that equations (3.2) and (3.4) do reproduce the Randall-Sundrum conditions $\eta_0 = -\eta_2 = 1$ or equivalently $\kappa^2 \rho_0 = -\kappa^2 \rho_2 = \sqrt{-6\Lambda}$,

in the constant brane tension ($\omega_0 = \omega_2 = -1$), non- Z_2 and time independent limit as expected. For the specific case where the equation of state of the reference brane is given by $p_0 = \omega_0 \eta_0$ with $\omega_0 = -1$, one finds from solving the Einstein equations that $a'/a = n'/n$. Equation (3.4) then becomes (using equation (3.1)),

$$\bar{p}_2 = \frac{\bar{\eta}_2 \left(\frac{f_2}{\bar{\eta}_2 a^4} + 3\mu\bar{\eta}_2 \right)}{3 \left(\frac{f_2}{\bar{\eta}_2 a^4} - \mu\bar{\eta}_2 \right)}. \quad (3.5)$$

Therefore, at times when the non- Z_2 symmetric nature of the second brane dominates i.e. when $f_2/\eta_2 a^4 \gg \mu\eta_2$ equation (3.5) implies that $\omega_2 = 1/3$. This corresponds to late time in a radiation dominated universe. Alternatively, if the second brane is approximately Z_2 symmetric such that $f_2/\eta_2 a^4 \ll \mu\eta_2$ then equation (3.5) predicts that $\omega_2 = -1$, which corresponds to the second brane possessing a constant brane tension. Another interesting case where equilibrium positions exist is when both branes are tuned, Z_2 -symmetric and possess realistic cosmologies. Here, equilibrium positions exist provided both branes have the same equation of state. This situation is discussed fully in section 5.4. The nature of the stability of the above equilibrium solutions will be examined in the next section.

3.2 Radion Fluctuations

In order to examine the nature of the equilibrium solutions found above it is necessary to linearise the radion equations (2.21) and (2.25) around the equilibrium points. Setting $\dot{\mathcal{R}} = \delta\dot{\mathcal{R}}$, $\ddot{\mathcal{R}} = \delta\ddot{\mathcal{R}}$, $\eta_2 = \bar{\eta}_2 + \delta\eta_2$ and $p_2 = \bar{p}_2 + \delta p_2$ results in,

$$\delta \left(\frac{a'}{a} \right) + \frac{\dot{a}}{an} \frac{\delta\dot{\mathcal{R}}}{n} = \left(\mu - \frac{f_2}{\bar{\eta}_2^2 a^4} \right) \delta\eta_2 - \frac{4f_2}{\bar{\eta}_2^2 a^5} \delta a, \quad (3.6)$$

$$\delta \left(\frac{n'}{n} \right) + \frac{\delta\ddot{\mathcal{R}}}{n^2} - \frac{\dot{n}}{n^2} \frac{\delta\dot{\mathcal{R}}}{n} = -2\mu\delta\eta_2 - 3\mu\delta p_2 + \frac{3f_2\bar{p}_2}{\bar{\eta}_2^2 a^4} \left(\frac{\delta p_2}{\bar{p}_2} - 2\frac{\delta\eta_2}{\bar{\eta}_2} - 4\frac{\delta a}{a} \right). \quad (3.7)$$

In order to evaluate the first term in each of the equations (3.6) and (3.7) it is now assumed that the reference brane respects Z_2 symmetry and possesses no Weyl tensor component. Therefore $f_0 = 0$ and $\mathcal{C} = 0$ or in terms of the bulk Schwarzschild masses on either side of the reference brane: $\mathcal{C}_0 = \mathcal{C}_1 = 0$. Since $\mathcal{C}_2 \neq 0$ the second brane will not in general be Z_2 symmetric. This means that the metric components, which are obtained by solving Einstein's equations (2.2), now look like [51],

$$a(t, y) = a_0(\cosh \mu y - \eta_0 \sinh \mu y), \quad (3.8)$$

$$n(t, y) = \cosh \mu y - \tilde{\eta}_0 \sinh \mu y, \quad (3.9)$$

where $\tilde{\eta}_0 = \eta_0 + \dot{\eta}_0/H_0$. It can now be explicitly shown using equations (3.1), (3.3), (3.8) and (3.9) that,

$$\delta\left(\frac{a'}{a}\right) = \left[\mu^2 - \left(\mu\bar{\eta}_2 + \frac{f_2}{\bar{\eta}_2 a^4}\right)^2\right]\delta\mathcal{R} \equiv m_a^2\delta\mathcal{R}, \quad (3.10)$$

$$\delta\left(\frac{n'}{n}\right) = \left[\mu^2 - \left(-2\mu\bar{\eta}_2 - 3\mu\bar{p}_2 + \frac{3f_2\bar{p}_2}{\bar{\eta}_2^2 a^4}\right)^2\right]\delta\mathcal{R} \equiv m_n^2\delta\mathcal{R}, \quad (3.11)$$

and defining the function $\bar{g}(t, \mathcal{R})$ as,

$$\bar{g}(t, \mathcal{R}) = \mu\bar{\eta}_2 + \frac{f_2}{\bar{\eta}_2 a^4}, \quad (3.12)$$

equations (3.6) and (3.7) can be rewritten in the following form:

$$\mathcal{H}_2 \frac{\delta\dot{\mathcal{R}}}{n} + (m_a^2 + 4\bar{g}(\bar{g} - \mu\bar{\eta}_2))\delta\mathcal{R} = (2\mu - \frac{\bar{g}}{\bar{\eta}_2})\delta\eta_2, \quad (3.13)$$

$$\begin{aligned} \frac{\delta\ddot{\mathcal{R}}}{n^2} - \frac{\dot{n}}{n^2} \frac{\delta\dot{\mathcal{R}}}{n} + (m_n^2 + 12\frac{\bar{p}_2}{\bar{\eta}_2}\bar{g}(\bar{g} - \mu\bar{\eta}_2))\delta\mathcal{R} = \\ -2(\mu + 3\frac{\bar{p}_2}{\bar{\eta}_2}(\bar{g} - \mu\bar{\eta}_2))\delta\eta_2 - 3(2\mu - \frac{\bar{g}}{\bar{\eta}_2})\delta p_2, \end{aligned} \quad (3.14)$$

where the fact that $\delta a/a = (a'/a)\delta\mathcal{R}$ and that $(\dot{a}/an)\delta\dot{\mathcal{R}} = \mathcal{H}_2\delta\dot{\mathcal{R}}$ to first order, has been used. Thus, it can be seen that in general the radion fluctuations are entangled with the matter fluctuations and that solving the linearised equations becomes non-trivial. It is possible however, when the fluctuations are of the adiabatic type, $\delta p_2 = c_2^2\delta\eta_2$ to take the appropriate linear combination of equations (3.13) and (3.14) to obtain an equation that depends upon the radion fluctuations only. Examining initially the case where $\omega_0 = -1$, $p_2 = \omega_2\eta_2$ and therefore $c_2^2 = \omega_2$, and where the second brane is dominated by its lack of Z_2 symmetry such that $f_2/\eta_2 a^4 \gg \mu\eta_2$, one finds that equations (3.13) and (3.14) combine to give,

$$\delta\mathcal{R}_{,\tau\tau} - 3\omega_2\mathcal{H}_2\delta\mathcal{R}_{,\tau} + 3\omega_2(1 - 3\omega_2)\frac{f_2^2}{\eta_2^2 a^8}\delta\mathcal{R} = 0, \quad (3.15)$$

where only the dominant terms have been kept. τ is defined as before as the time experienced by an observer on the second brane. The matter fluctuations are now related to the radion fluctuations by the formula:

$$\delta\eta_2 = -\frac{\eta_2^2 a^4}{f_2} \left(\mathcal{H}_2\delta\mathcal{R}_{,\tau} + \frac{3f_2^2}{\eta_2^2 a^8}\delta\mathcal{R} \right). \quad (3.16)$$

Equation (3.15) can now be solved for a radiation dominated second brane where $\omega_2 = 1/3$. Using the approximation $\mathcal{H}_2 \simeq -f_2/\eta_2 a^4$ and remembering that in this situation $f_2 < 0$, one obtains the expression,

$$\delta\mathcal{R} = \delta\mathcal{R}_i e^{-f_2(\tau-\tau_i)/\eta_2 a^4} + \left(4\delta\mathcal{R}_i + \frac{\eta_2 a^4}{\eta_{2i} f_2} \delta\eta_{2i} \right) (1 - e^{-f_2(\tau-\tau_i)/\eta_2 a^4}), \quad (3.17)$$

where the subscript i means the value of the quantity at τ_i . Therefore it is clear that when there is a constant brane tension on the reference brane and a highly non- Z_2 symmetric second brane such as is the case suggested by Randall and Lyken [58], an equilibrium solution exists only when the second brane is radiation dominated, however this solution for the interbrane distance or cosmological radion is inherently unstable.

Another case of interest is when both branes are Z_2 symmetric and as discussed in the previous section if $\omega_0 = -1$ then an equilibrium solution exists only when $\omega_2 = -1$ also. Adding three times equation (3.13) to equation (3.14) and neglecting all f_2 terms results in,

$$\delta\mathcal{R}_{,\tau\tau} + 3\mathcal{H}_2\delta\mathcal{R}_{,\tau} + 4\mu^2(1-\eta_2^2)\delta\mathcal{R} = -\frac{\kappa^2}{6}\delta T_2, \quad (3.18)$$

where δT_2 is the linear perturbation of the trace of the matter energy-momentum tensor [51] and is given by,

$$\frac{\kappa^2}{6}\delta T_2 = -\mu\delta\eta_2 + 3\mu\delta p_2. \quad (3.19)$$

It can then be seen that equation (3.18) is just the equation of motion of a scalar field coupled to the perturbed trace of the matter energy-momentum tensor. This therefore confirms previous results found in [47–49]. The coefficient of the $\delta\mathcal{R}$ term in equation (3.18) demonstrates that when both branes are dS_4 implying that both $\eta_0^2 > 1$ and $\eta_2^2 > 1$, small perturbations around the equilibrium radion position are unstable. Alternatively, if both branes are AdS_4 and therefore $\eta_0^2 < 1$ and $\eta_2^2 < 1$ then the perturbations are stable. It is again possible in this case to eliminate the matter fluctuation terms by taking a suitable linear combination of equations (3.13) and (3.14). For a general ω_2 and keeping $\omega_0 = -1$ this leads to,

$$\delta\mathcal{R}_{,\tau\tau} + (2 + 3\omega_2)\mathcal{H}_2\delta\mathcal{R}_{,\tau} + 3\mu^2(1 + \omega_2)(1 - (2 + 3\omega_2)\eta_2^2)\delta\mathcal{R} = 0. \quad (3.20)$$

The matter fluctuations are now related to the radion fluctuations by,

$$\mu\delta\eta_2 = \mathcal{H}_2\delta\mathcal{R}_{,\tau} + \mu^2(1 - \eta_2^2)\delta\mathcal{R}. \quad (3.21)$$

Assuming that both branes are dS_4 , that $\omega_2 = -1$ and that $k = \mathcal{C}_1 = \mathcal{C}_2 = 0$ ensures that \mathcal{H}_2 is time independent and given by $\mathcal{H}_2 = \mu^2(\eta_2^2 - 1)$. It is therefore possible to solve equation (3.20) for $\mathcal{R}(\tau)$ giving,

$$\delta\mathcal{R} = \delta\mathcal{R}_i e^{\mathcal{H}_2(\tau - \tau_i)} - \frac{1}{\mathcal{H}_2^2}\delta\eta_{2i}(1 - e^{\mathcal{H}_2(\tau - \tau_i)}). \quad (3.22)$$

where as before $\delta\mathcal{R}_i$ and $\delta\eta_{2i}$ are the initial radion fluctuations and energy density fluctuations at time τ_i . This describes the behaviour near the equilibrium position

and demonstrates explicitly the unstable nature of the radion for two dS_4 branes and will be compared with several numerical solutions discussed in the next section [51]. If instead, both branes are assumed to be AdS_4 then it is not possible for \mathcal{H}_2 to be time independent and either k , \mathcal{C}_1 or \mathcal{C}_2 must be non-zero. This leads to a recollapsing universe on the second brane. Again taking $\omega_2 = -1$ allows equation (3.20) to be written as simply,

$$\delta\mathcal{R}_{,\tau\tau} = \mathcal{H}_2 \delta\mathcal{R}_{,\tau}. \quad (3.23)$$

Examining equations (3.21) and (3.23) it can be seen that the radion will only exhibit a pseudo-stable behaviour in that \mathcal{R} will initially accelerate toward the equilibrium position but will only decelerate once \mathcal{H}_2 has changed from positive to negative. In the section 5 the analytic results discussed here will be compared with complete numerical solutions of the radion equation for several cases of interest. Before this we go on in the next section to discuss the cosmological radion in terms of the static bulk perspective, in order to obtain an intuitive understanding of some of the numerical results that are presented in section 5.

4. The Two Brane Scenario: a Bulk Perspective

In the previous section we linearised the radion equations in order to determine the stability of several equilibrium positions, and therefore to obtain some idea of the behaviour of the radion in several cases of interest. Here we go on to briefly examine the two brane model from the perspective of a bulk observer. This will allow us to show that the evolution of the radion can be thought of as a competition between the expansion rates of the two branes, and this will help determine in which cases the branes collide or move apart. These predictions will be shown to be in agreement with the numerical results presented in the next section.

We are considering purely five-dimensional spacetimes that respect the symmetries of three-dimensional homogeneity and isotropy, and that only possess a five dimensional cosmological constant in the bulk. It was first shown by [15] that a modified version of Birkhoff's theorem implies that such spacetimes are in fact static. The bulk metric takes the general form:

$$ds^2 = -\phi(R) dT^2 + R^2 \gamma_{ij} dx^i dx^j + \phi(R)^{-1} dR^2, \quad (4.1)$$

where we have defined the function $\phi(R)$ as,

$$\phi(R) = \left(\mu^2 R^2 + k - \frac{\mathcal{C}_i}{R^2} \right), \quad (4.2)$$

and as before k takes the values 0, -1 , or 1 for flat, closed or open geometries, and γ_{ij} is the corresponding metric on the unit plane, hyperboloid or sphere. T is the time experienced by a bulk observer, and R the coordinate of the fifth dimension,

while as before the subscript i is either 0, 1 or 2, corresponding to the bulk regions to the left, between or to the right of the branes.

The simple form of these bulk solutions is counter-balanced by the fact that in these coordinates both the 3-branes cannot (except in trivial cases) be stationary, and in general move through the bulk spacetime. In fact in the non- Z_2 symmetric case, the brane moves through two different bulk spacetimes, with for example, two different bulk masses either side of the brane. If the reference brane has a trajectory given by $R = R_0(T)$, then the induced metric on it is given by,

$$ds^2 = -dt^2 + R_0^2(t)\gamma_{ij}dx^i dx^j, \quad (4.3)$$

where again t is the time experienced by an observer on the reference brane and is related to the bulk time T by,

$$dt^2 = \left[\phi(R_0) - \frac{1}{\phi(R_0)} \left(\frac{dR_0}{dT} \right)^2 \right] dT^2. \quad (4.4)$$

Hence from equation (4.3) one can see that the apparent expansion of the brane as seen by such an observer is in fact caused by the brane's motion through the bulk [15]. One can therefore deduce that the brane's position $R_0(t)$ in these bulk coordinates is proportional to the scale factor $a_0(t)$ of the brane used throughout the rest of this paper. Thus the trajectory of the brane, at least in terms of t , can be found from its Friedmann equation (2.10). In order to compare the trajectories of two such branes we need to solve for R_0 in terms of the bulk time T . Equation (4.4) can be rearranged to give,

$$\frac{dt}{dT} = \frac{\phi(R_0)}{(\phi(R_0) + H_0^2 R_0^2)^{1/2}}, \quad (4.5)$$

where H_0 is again the Hubble parameter on the reference brane given by $H_0 R_0 = dR_0/dt$. This can be combined with equation (4.5) to show that the brane's motion in terms of T satisfies,

$$\frac{dR_0}{dT} = \frac{dR_0}{dt} \frac{dt}{dT} = \frac{H_0 R_0 \phi(R_0)}{(\phi(R_0) + H_0^2 R_0^2)^{1/2}}. \quad (4.6)$$

By replacing R_0 and H_0 with R_2 and H_2 , one can obtain the equivalent equation for the second brane,

$$\frac{dR_2}{dT} = \frac{\mathcal{H}_2 R_2 \phi(R_2)}{(\phi(R_2) + \mathcal{H}_2^2 R_2^2)^{1/2}}. \quad (4.7)$$

It must be remembered that here $R_0(T)$ and $R_2(T)$ correspond to the positions of the branes in the static spacetime that lies between them. The interbrane distance in terms of these bulk coordinates is now just given by $R = R_0 - R_2$, where $R_0(T)$

and $R_2(T)$ can be found by solving equations (4.6) and (4.7). For example, if we assume the reference brane is Z_2 symmetric, equation (4.6) becomes,

$$\frac{dR_0}{dT} = \frac{H_0\phi(R_0)}{\mu|\eta_0|}, \quad (4.8)$$

which assuming that $k = \mathcal{C}_0 = \mathcal{C}_1 = 0$, reduces to,

$$\frac{dR_0}{dT} = \mu^2 R_0^2 \sqrt{1 - \frac{1}{\eta_0^2}}. \quad (4.9)$$

If the reference brane possesses a constant dimensionless energy density η_0 , then this can be solved to give the position of the reference brane to be,

$$R_0(T) = \left[\frac{1}{R_{0i}} - \mu^2 \sqrt{1 - \frac{1}{\eta_0^2}} (T - T_i) \right]^{-1}, \quad (4.10)$$

where R_{0i} is the position of the brane at time T_i . If we make similar assumptions for the second brane, in that it is also Z_2 symmetric ($\mathcal{C}_2 = 0$) and that η_2 is constant, then $R_2(T)$ is found to be of the same form,

$$R_2(T) = \left[\frac{1}{R_{2i}} - \mu^2 \sqrt{1 - \frac{1}{\eta_2^2}} (T - T_i) \right]^{-1}, \quad (4.11)$$

where again $R_{2i} = R_2(T_i)$. Now taking the convention $R_{0i} > R_{2i}$, which here corresponds to the reference brane being of positive tension and the second brane being of negative tension, we can ask under what conditions the branes collide. $R_0(T) = R_2(T)$ occurs at a time T_c given by,

$$T_c = \frac{1/R_{2i} - 1/R_{0i}}{\mu(\mathcal{H}_2/|\eta_2| - H_0/\eta_0)} + T_i \quad (4.12)$$

however the collision must occur for positive R_0 and R_2 , and $R_0(T) > 0$ only when $T < T_\infty$, where T_∞ is given by,

$$T_\infty = \frac{\eta_0}{\mu H_0 R_{0i}} + T_i. \quad (4.13)$$

Therefore a collision will occur only if $T_c < T_\infty$, which implies the condition,

$$\frac{R_{2i}}{R_{0i}} > \frac{H_0|\eta_2|}{\mathcal{H}_2\eta_0} = \left(\frac{1 - 1/\eta_0^2}{1 - 1/\eta_2^2} \right)^{1/2}. \quad (4.14)$$

So using the bulk perspective we have found the requirement for brane collision to occur in terms of the branes' energy densities and initial positions, in the constant brane tension case. This condition (4.14), is in fact a specific case of the brane based

equilibrium conditions (3.1) and (3.3). This will be discussed further in section 5.1 where we look at the constant brane tension case from the brane perspective, and where the equivalence of these conditions will be demonstrated explicitly.

For more realistic cases such as where both branes possess a cosmologically evolving energy density, the bulk perspective is less useful. It is, in general not possible to analytically solve equations (4.6) and (4.7) in order to obtain the trajectories of the branes and hence it is difficult to determine whether the branes will collide. In addition one cannot easily determine the conditions for equilibrium of the interbrane distance; however it must be noted that this equilibrium is in some sense a creation of the brane coordinates themselves. One inevitably has to turn to numerical methods, and due to this in the next section we numerically integrate the radion equation, formulated in terms of the brane coordinates. Note that in the brane based perspective, all the information about the branes' relative positions contained in equations (4.6) and (4.7) is described by one equation (2.21), the non-linear equation for the radion. Hence we only need to numerically solve one non-linear equation.

The bulk perspective however, can still provide intuitive insight into the results obtained in the following chapter. Thinking of the evolution of the interbrane distance as a competition between the branes' expansion rates helps explain the rich variety of behaviour that is found.

5. Numerical Analysis of the Non-Linear Radion

Previously, the equations for the cosmological radion have been derived and analysed. The equilibrium points have been identified and the stability of these solutions examined. Unfortunately, due to the highly non-linear nature of the radion equation there is only so much one can learn from analytical methods. Hence, in this section, the radion equations are solved numerically in several different cases in order to determine the non-perturbative aspects of the cosmological radion. The subtle numerical problems involved are discussed, and the results are interpreted using the bulk perspective described in the previous section. We first review and then extend some of the results on dS and AdS branes presented in [51].

5.1 Constant Tensions on Both Branes: The dS Case

If the assumptions are made that the reference brane is dS_4 or equivalently that $\eta_0 > 1$, and that $\mathcal{C}_0 = \mathcal{C}_1 = 0$ implying that the brane possesses Z_2 symmetry and has no Weyl tensor component, then the brane based metric components can be written in the simple form,

$$a(t, y) = a_0(t)(\cosh \mu y - \eta_0 \sinh \mu y) = a_0(t)\sqrt{\eta_0^2 - 1} \sinh \mu(y_h - |y|), \quad (5.1)$$

$$n(y) = (\cosh \mu y - \eta_0 \sinh \mu y) = \sqrt{\eta_0^2 - 1} \sinh \mu(y_h - |y|), \quad (5.2)$$

where y_h denotes the position of the coordinate singularity defined by $a(t, y) = 0$ and is given by,

$$\mu y_h = \tanh^{-1} \frac{1}{\eta_0}. \quad (5.3)$$

Replacing the metric components (5.1) and (5.2) into equations (3.1) and (3.3) to determine the equilibrium radion position denoted by y_e , one finds as stated before that the equation of state on the second brane must be of the form $p_2 = -\eta_2$ and that,

$$\mu y_e = \tanh^{-1} \frac{1}{\eta_0} + \tanh^{-1} \frac{1}{\eta_2}. \quad (5.4)$$

In order to numerically solve the radion equation here and in the following sections, it is useful to convert to the following dimensionless variables,

$$z = \mu \mathcal{R}, \quad z_{h,e} = \mu y_{h,e} \quad s = \mu t, \quad h_0 = \frac{H_0}{\mu}, \quad h_2 = \frac{\mathcal{H}_2}{\mu}. \quad (5.5)$$

Replacing the expressions for the metric coefficients (5.1) and (5.2) into the radion equation (2.21) and changing to the above dimensionless variables gives,

$$z_{,s} = h_0 \sinh(z_h - z) \frac{\cosh(z_h - z) + h_2 \eta_2 \sinh^2(z_h - z)}{1 + \eta_2^2 \sinh^2(z_h - z)}, \quad (5.6)$$

where it should be remembered that here $h_0^2 = \eta_0^2 - 1$, and setting $k = 0$ gives $h_2^2 = \eta_2^2 - 1$. Since the energy densities of both branes are by assumption constant in time, the evolution of the radion is completely embodied by equation (5.6) and all that is required to solve it is the initial radion position denoted by z_i . Numerical solutions were generated for two different situations, each with varying initial radion positions and the results are shown in figures 1 and 2. Figure 1 corresponds to the case where $\eta_0 = 1.5$ and $\eta_2 = -3$ and shows the trajectories of six different initial radion positions distributed around the equilibrium position z_e . Equations (5.3) and (5.4) show that for this choice of brane tensions the equilibrium and singularity positions are $z_e \simeq 0.458$ and $z_h \simeq 0.805$ which correspond to the first and second dotted lines of figure 1. As expected from the above analysis, the equilibrium position is found to be unstable: initial positions satisfying $z_i < z_e$ lead to the second brane colliding with the reference brane; as opposed to initial positions satisfying $z_i > z_e$ when the second brane asymptotically ‘freezes out’ at z_h as seen by an observer on the reference brane. Consideration of equations (5.6) and (2.13) demonstrates that crossing z_h only takes a finite amount of proper time on the second brane.

The situation where $\eta_0 = 1.5$ and $\eta_2 = -1.3$ was investigated and the results are shown in figure 2. Referring to the expression for the equilibrium position given by equation (5.4), it can be seen that if both branes are dS with for example η_0 positive and η_2 negative, then $|\eta_0| > |\eta_2|$ implies that $z_e < 0$. Due to the Z_2 symmetry this means that there is no physical equilibrium position for the radion in this setup. This

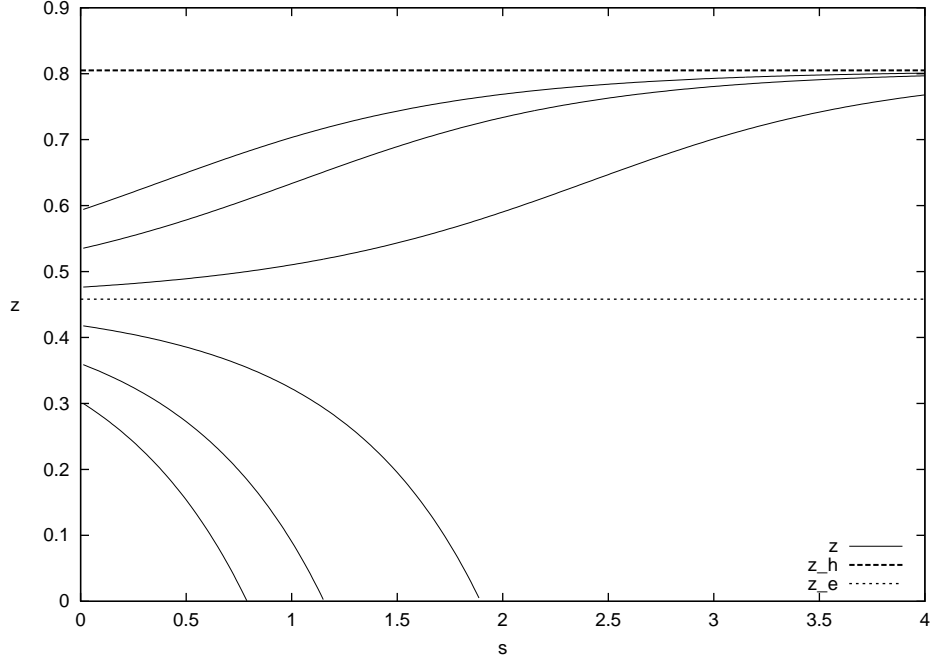


Figure 1: Both branes are dS and $\eta_0 = 1.5$, $\eta_2 = -3$.

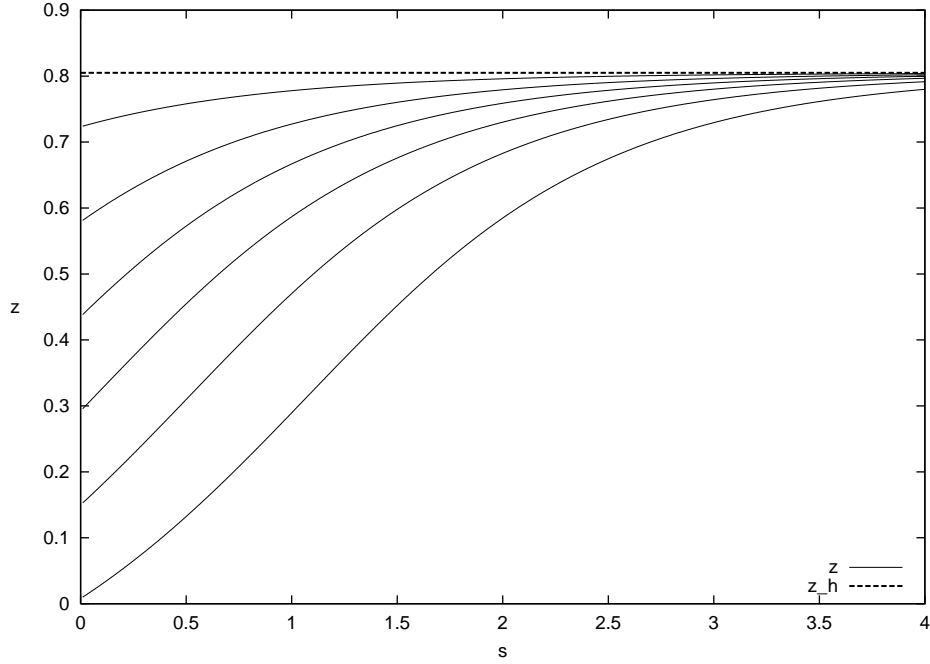


Figure 2: Both branes are dS and $\eta_0 = 1.5$, $\eta_2 = -1.3$.

is confirmed by figure 2 where every choice of initial position for the radion leads to the second brane freezing out at z_h , where $z_h \simeq 0.805$ as in the previous example. This behaviour can be better understood from a bulk observer's perspective. In the bulk coordinate system, both branes are moving through a static background. Their

trajectories from this point of view are solely dependent upon their expansion rates or Hubble constants; increasing the magnitude of η_2 increases \mathcal{H}_2 and hence improves the chances of the second brane ‘catching’ the reference brane. Alternatively, if as in figure 2, the magnitude of η_0 is greater than that of η_2 , it will be impossible for the branes to collide. In section 4 we used the bulk perspective to solve for the trajectories of two constant tension dS branes, and derived the condition for collision (4.14): $R_{2i}/R_{0i} > H_0|\eta_2|/\mathcal{H}_2\eta_0$. This can be rewritten using the fact that R_0 and R_2 are proportional to $a_0(t)$ and $a(t, z)$ respectively, which implies,

$$\frac{\alpha a(t, z)}{a_0(t)} > \frac{H_0|\eta_2|}{\mathcal{H}_2\eta_0}, \quad (5.7)$$

where we have scaled the three spatial dimensions x^i so that $R_0 = a_0$ and therefore $R_2 = \alpha a(t, z)$ where α is some positive constant discussed below. Again using the explicit expression for the metric element $a(t, z)$ given by equation (5.1), this becomes,

$$\sinh(z_h - z) > \frac{|\eta_2|}{\alpha\eta_0} \frac{1}{(\eta_2^2 - 1)^{1/2}}. \quad (5.8)$$

Now assuming that $\alpha = |\eta_2|/\eta_0$, one finds that the branes will collide if the initial z satisfies,

$$\tanh(z_h - z) > \frac{1}{|\eta_2|}, \quad (5.9)$$

from which one can recover the equilibrium condition equation (5.4), derived from the brane perspective. One can check that α does indeed take the required form, as it must in order for the two perspectives to agree, by checking the explicit transformation between bulk and brane coordinates as detailed in [59].

The bulk perspective is very useful for understanding the behaviour of the non-perturbative cosmological radion and will be discussed further. It must be stressed that a time independent bulk solution will not be possible for scenarios that contain more general matter in the bulk, such as for example scalar fields or dilaton fields. This is another reason for using brane based coordinates to examine the radion.

5.2 Constant Tensions on Both Branes: The AdS Case

In this section the radion dynamics are investigated for the case of two AdS_4 branes, that is two branes such that $|\eta_0| < 1$ and $|\eta_2| < 1$. Again it is assumed that both branes are Z_2 -symmetric and that they possess no Weyl tensor component such that $\mathcal{C}_0 = \mathcal{C}_1 = \mathcal{C}_2 = 0$. This implies that one must take $k = -1$ to get a consistent solution to the Einstein equations. The metric coefficients take the form,

$$a(t, y) = a_0(t)(\cosh \mu y - \eta_0 \sinh \mu y) = a_0(t)\sqrt{1 - \eta_0^2} \cosh(|z| - z_m), \quad (5.10)$$

$$n(y) = (\cosh \mu y - \eta_0 \sinh \mu y) = \sqrt{1 - \eta_0^2} \cosh \mu(|z| - z_m), \quad (5.11)$$

where the dimensionless coordinate $z = \mu y$ has been used as before, and z_m which corresponds to the minimum of $a(s, z)$ and $n(s, z)$ is given by,

$$z_m = \tanh^{-1} \eta_0. \quad (5.12)$$

From equation (2.20) it is seen that since $a(s, z)$ possesses a minimum, it is possible to have two positive tension branes in a compactified five-dimensional spacetime [60]. The radion equation (2.21) now looks like,

$$z_{,s} = K_0^2 \cosh(z - z_m) \times \quad (5.13)$$

$$\frac{h_0 \sinh(z - z_m) \pm \eta_2 \cosh(z - z_m) \sqrt{h_0^2 + K_0^2(1 - K_2^2 \cosh^2(z - z_m))}}{h_0^2 + \eta_2^2 K_0^2 \cosh^2(z - z_m)}, \quad (5.14)$$

where $K_0 = \sqrt{1 - \eta_0^2}$ and $K_2 = \sqrt{1 - \eta_2^2}$. Using equation (2.10) the dimensionless Hubble parameter on the reference brane is now given by,

$$h_0 = -K_0^2 + \frac{1}{a_0^2}, \quad (5.15)$$

which can be solved to give $a_0(t) = \sin(K_0 s)/\mu K_0$ and therefore $h_0 = K_0/\tan(K_0 s)$ which is all that is needed in order to numerically solve equation (5.13). It must be noted that, when generating a solution to equation (5.13) one must be careful to ensure that the sign of the square root is changed whenever the argument of the square root vanishes i.e. when $h_2 = 0$. This has to be the case and can be seen on examination of equation (2.21). The equilibrium position can now be shown to be,

$$z_e = \tanh^{-1} \eta_0 + \tanh^{-1} \eta_2. \quad (5.16)$$

Figures 3 and 4 show the radion trajectories that were numerically generated for two distinct situations. The first case that was examined, corresponding to figure 3, was where $\eta_0 = 0.6$ and $\eta_2 = 0.6$ which implies choosing the positive root in equation (5.13). The initial radion positions are distributed about the equilibrium point $z_e = 1.39$ which is represented by the dotted line. As expected from the analysis in section 3.2, it is seen that the the radion displays a pseudo-stable behaviour - initially the interbrane distance accelerates toward z_e , but it only begins to decelerate once $h_2 = 0$ which does not necessarily correspond to z crossing the equilibrium point. It is this fact combined with the warped nature of $n(s, z)$ that causes different trajectories to reach $h_2 = 0$ at different points and hence leads to the asymmetrical (in the z direction) appearance of figure 3. Figure 4 shows several radion trajectories when the second brane possesses a negative tension. Specifically, the brane tensions have been chosen to be $\eta_0 = 0.9$ and $\eta_2 = -0.9$ which implies that $z_e = 0$. As expected, this leads to the second brane colliding with the first for any given initial radion position.

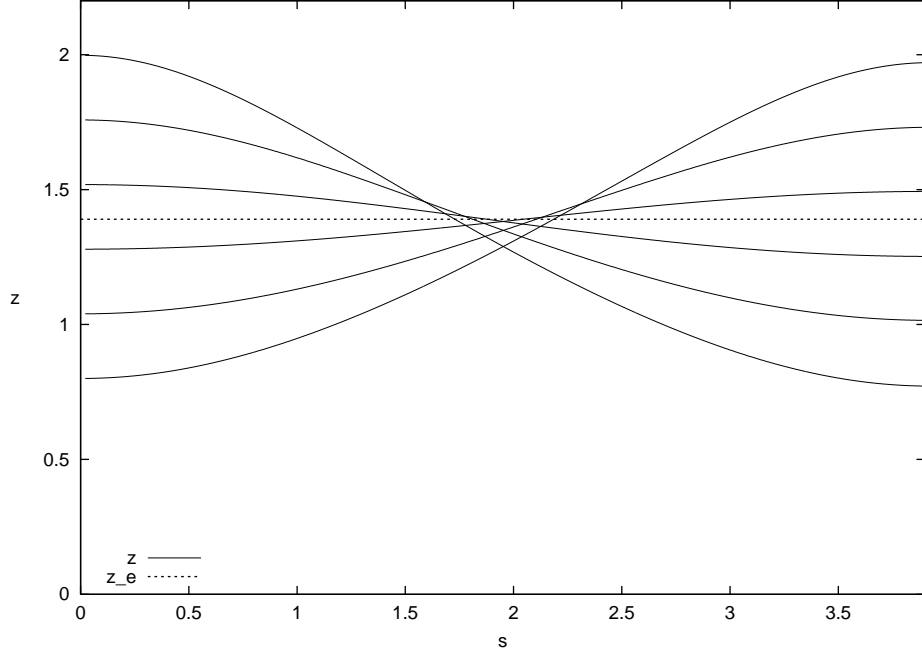


Figure 3: Both branes are AdS and $\eta_0 = 0.6, \eta_2 = 0.6$.

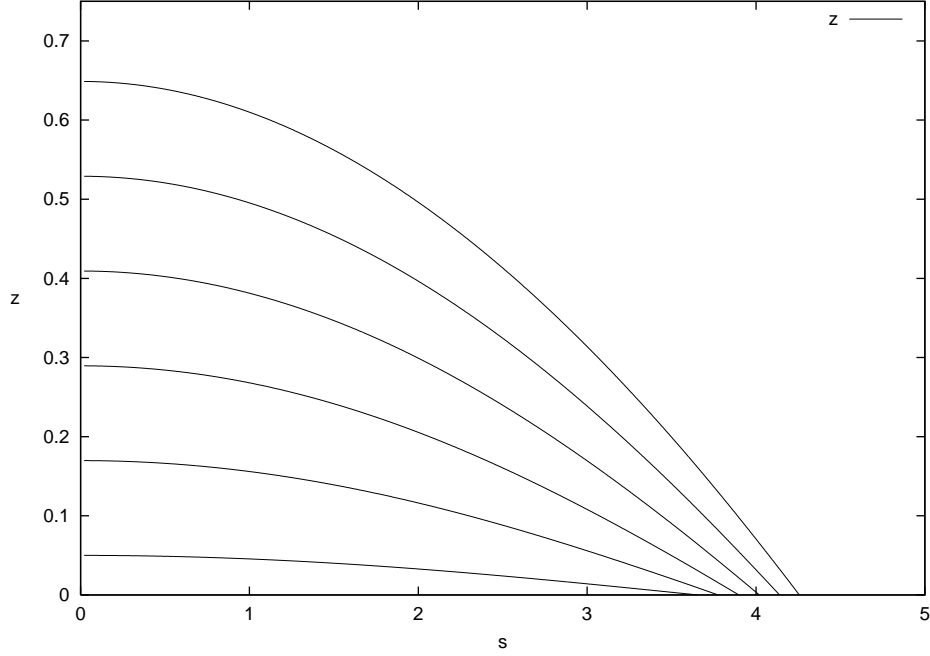


Figure 4: Both branes are AdS and $\eta_0 = 0.9, \eta_2 = -0.9$.

5.3 Radiation and Matter on the Reference Brane

We now turn to a more interesting situation where there exists a realistic cosmology on the reference brane. We assume that the dimensionless energy density η_0 can be decomposed into two parts [61]: a constant brane tension and a ‘physical’ energy

density denoted by η_{0p} such that,

$$\eta_0 = 1 + \eta_{0p}, \quad (5.17)$$

and similarly the dimensionless pressure is decomposed into,

$$p_0 = -1 + p_{0p}. \quad (5.18)$$

This is done as before to ensure that the reference brane undergoes standard cosmology at late times or more specifically that the Friedmann equation has the form,

$$H_0^2 = \mu^2 \eta_{0p}^2 + 2\mu^2 \eta_{0p}, \quad (5.19)$$

so that $H_0^2 \propto \eta_{0p}$ at late times, where we have assumed that both branes are Z_2 symmetric, possess no Weyl tensor component and that $k = 0$. If the reference brane now has an equation of state $p_{0p} = \omega_0 \eta_{0p}$, equations (3.8) and (3.9) show that the metric coefficients in terms of s and z will now be given by,

$$a(s, z) = a_0(\cosh z - (1 + \eta_{0p}) \sinh z), \quad (5.20)$$

$$n(s, z) = \cosh z - (1 - (2 + 3\omega_0))\eta_{0p} \sinh z, \quad (5.21)$$

where now both a_0 and η_{0p} are functions of the dimensionless time parameter s . The Friedmann equation (5.19) can now be solved for η_{0p} using the brane energy conservation equation,

$$\dot{\eta}_0 = -3H_0(\eta_0 + p_0), \quad (5.22)$$

which leads to,

$$\eta_{0p}(s) = \frac{1}{\frac{1}{2}q_0^2 s^2 + q_0 s}, \quad (5.23)$$

where we have defined $q_0 = 3(1 + \omega_0)$. Using this expression for $\eta_{0p}(s)$, the position of the coordinate singularity defined by $a(s, z_h) = 0$ is now given by the simple formula,

$$z_h = \tanh^{-1} \left(\frac{1}{1 + \eta_{0p}} \right) = \ln(q_0 s + 1). \quad (5.24)$$

We now assume that the reference brane is in a radiation dominated phase in that $\omega_0 = 1/3$, $q_0 = 4$ and therefore $\eta_{0p} \propto a_0^{-4}$ and that the second brane has a constant brane tension. It is now possible, using equations (5.17), (5.19), (5.20), (5.21) and (5.23) to numerically solve the radion equation (2.21). This was done using several different initial conditions and the results are shown in figure 5, along with the time dependent position of z_h denoted by the dotted line and given by equation (5.24). The six initial radion positions were distributed evenly between $z = 0$ and $z = z_h$. In figure 5 the radion positions are graphed against time for a second brane tension $\eta_2 = -1.2$, and as can be seen, all the radion trajectories lead to the two branes

colliding. Trajectories that initially begin closer to z_h take longer to return to the reference brane as would be expected.

Similar results were also obtained for various values of the second brane tension. It was found that the radion trajectories exhibit the same general behaviour, but the second brane would return and collide with the reference brane sooner/later if the magnitude of η_2 was greater/lesser. An interesting case is shown in figure 6, where the reference brane is again radiation dominated, however the second brane is now a tuned or critical brane with $\eta_0 = -1$. The initial radion positions were spread between $z = 0$ and $z = 0.95z_h$ at $s = 0.05$ and the coordinate singularity moves away from the reference brane as before with $z_h = \ln(4s + 1)$. In this situation the second brane does not return and collide with the reference brane, instead the interbrane distance increases with logarithmic behaviour similar to that of the coordinate singularity at z_h . If instead, it is assumed that the reference brane is in a matter dominated phase so that $\omega_0 = 0$, $q_0 = 3$ and therefore $\eta_{0p} \propto a_0^{-3}$, the dynamics of the interbrane distance will again be altered. The coordinate singularity is now at $z_h = \ln(3s + 1)$, and it was found that if the reference brane is matter dominated, the second brane takes longer to return.

This general radion behaviour when the reference brane has a more realistic cosmology can be understood in terms of the evolution of the brane energy densities and by viewing the situation from a bulk observers perspective. Initially η_{0p} is very large causing a_0 to increase rapidly ($a_0 \propto t^{1/4}$) and therefore the reference brane will be moving rapidly through the static bulk. The second brane on the other hand, only has a low constant energy density η_2 , and will begin by moving slowly, hence the interbrane distance will initially increase. At late times, η_{0p} tends to zero and the reference brane is much slower compared to the second brane which now has $a_2 \propto \exp \mathcal{H}_2 \tau$ and therefore the interbrane distance will decrease and eventually the branes will collide. This means that the time taken to collide is decreased by having a larger η_2 , in agreement with the above results. Alternatively, a matter dominated reference brane will maintain a higher ‘speed’ at late times and therefore will increase the time taken to collide. In figure 6, where the tuned second brane never returns, we have that $\mathcal{H}_2 = 0$, and therefore from the bulk observers point of view the second brane is stationary. Since our brane is expanding (albeit with a continually decreasing rate), it moves away from the second brane and hence the interbrane distance increases monotonically. All these features of the radion’s behaviour will be discussed further in the next section when we go on to investigate the time taken for the branes to collide.

5.3.1 The Time Taken for the Branes to Collide

In order to obtain an intuitive understanding of the nature of the radion it is useful to investigate explicitly s_{col} , the dimensionless time taken for the branes to collide. s_{col} depends on many variables including the initial conditions, the equations of

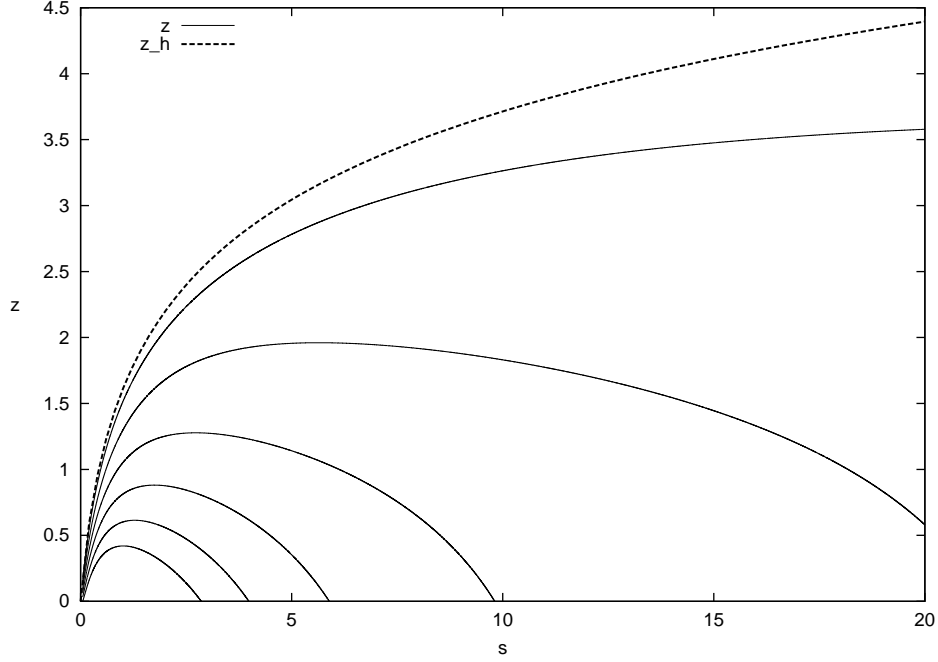


Figure 5: Radiation on the reference brane, $\eta_2 = -1.2$ on the second brane.

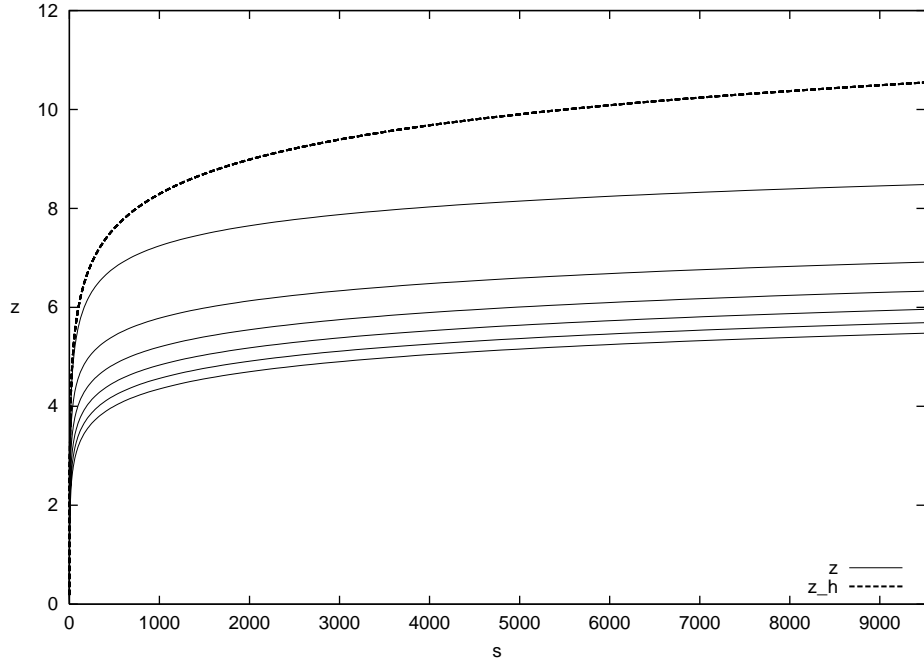


Figure 6: Radiation on the reference brane with a tuned ($\eta_2 = -1$) second brane.

state on either brane, the curvature and non- Z_2 symmetry of the branes and the proximity of the second brane to z_h . Here we have restricted our analysis to the situations presented in the previous section, where the reference brane possesses a realistic cosmology and the second brane has a constant negative brane tension.

The dependence of the collision time on the initial radion position z_i , when the reference brane is either radiation or matter dominated was numerically determined and the results are shown in figure 7. Firstly one can see that for all values of z_i/z_h , the branes collide much sooner for a radiation as opposed to a matter dominated reference brane. As discussed in the previous section, this is due to the matter dominated brane moving through the bulk with a greater ‘speed’ than that of the radiation dominated brane. Secondly, as the initial radion position increases, the collision time also increases, becoming singular as $z_i \rightarrow z_h$. The reason for this is that $a(s, z) \rightarrow 0$ at z_h , and therefore the initial scale factor of a brane close to z_h tends to zero also. Since the second brane has to expand from $a(s, z_i)$ to $a_0(s_{col})$ in order for collision to occur, a vanishing $a(s, z_i)$ implies $s_{col} \rightarrow \infty$.

The effect on the collision time of varying the tension of the second brane was investigated, and the results are shown in figure 8. A similar effect to figure 7 is seen in that radiation domination implies faster collision. In addition, it can be seen that increasing the magnitude of η_2 leads to the second brane moving faster, which in turn decreases s_{col} . As can be seen from figure 6, when $\eta_2 \rightarrow -1$ then $s_{col} \rightarrow \infty$, since the second brane is effectively stationary from the bulk point of view.

5.3.2 The Ekpyrotic and Cyclic Models

The above discussion of the collision times of the two branes highlights an important feature of the well known ekpyrotic and cyclic models. In the Ekpyrotic universe [62–67] it was proposed that the supposed beginning of our four-dimensional universe is in fact the result of a collision between a boundary and a bulk brane. Some of the energy of the collision would be converted into a hot big bang, and it was shown how one therefore no longer needs to incorporate inflation to explain the current observational data. The cyclic model [68, 69] went one step further: it was suggested that the big bang was due to the collision of two boundary branes, and that these branes would collide periodically every several trillion years. This would also explain the apparent accelerated expansion of the universe we observe today.

Both these models, however, needed to employ a potential between the branes in order for the brane collisions to occur at the correct times. Although these potentials were no more complex than those employed in the most recent inflationary theories, they were to say the least ad hoc. We have investigated throughout this paper the *natural* behaviour of cosmologically realistic branes moving in a five-dimensional *AdS* bulk, and as can be seen from the above results, we find that brane collisions (at least in this model) are commonplace. We also however, find that the collision time will not be significantly greater than the inverse five-dimensional Plank mass³ ($\simeq 30\text{TeV}$ or above), and therefore to delay brane collisions to the extent proposed in the Ekpyrotic and Cyclic models, a certain amount of fine tuning must be involved.

³A fact which can be inferred by dimensional analysis.

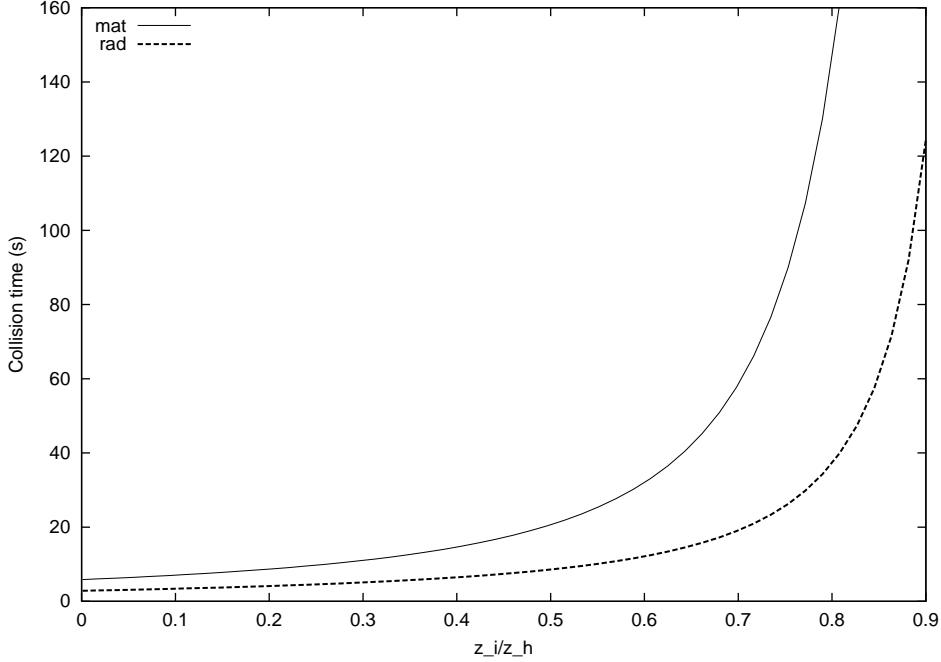


Figure 7: Time taken for branes to collide for varying initial z

Taking this point further, in general one might expect a five-dimensional model of this kind to contain a large number of branes, each containing different equations of state. These would necessarily be colliding over timescales of around M_5^{-1} , so in the case of a Cyclic universe the main question to answer would be why our brane has not hit anything else in the last thirteen billion years.

5.4 Radiation and Matter on Either Brane

We now turn to the interesting case where both branes are Z_2 symmetric and have time dependent energy densities $\eta_0(t)$ and $\eta_2(t)$. This allows us to investigate the behaviour of the radion when both branes are either radiation or matter dominated. There are, however, some subtleties as defining a time dependent energy density on the second brane leads in most cases to $\eta(t \rightarrow 0) = -\infty$, which could lead to some phenomenological problems.

Assuming that both brane energy densities can be decomposed into a constant brane tension component and a time dependent ‘physical’ component we have,

$$\eta_0 = 1 + \eta_{0p}, \quad p_0 = -1 + p_{0p}, \quad (5.25)$$

for the reference brane, and,

$$\eta_2 = -1 - \eta_{2p}, \quad p_2 = 1 - p_{2p}, \quad (5.26)$$

for the second brane where η_{2p} and p_{2p} are positive quantities. We have assumed that $\mathcal{C}_0 = \mathcal{C}_1 = \mathcal{C}_2 = k = 0$, and we want to obtain the Randall-Sundrum two brane

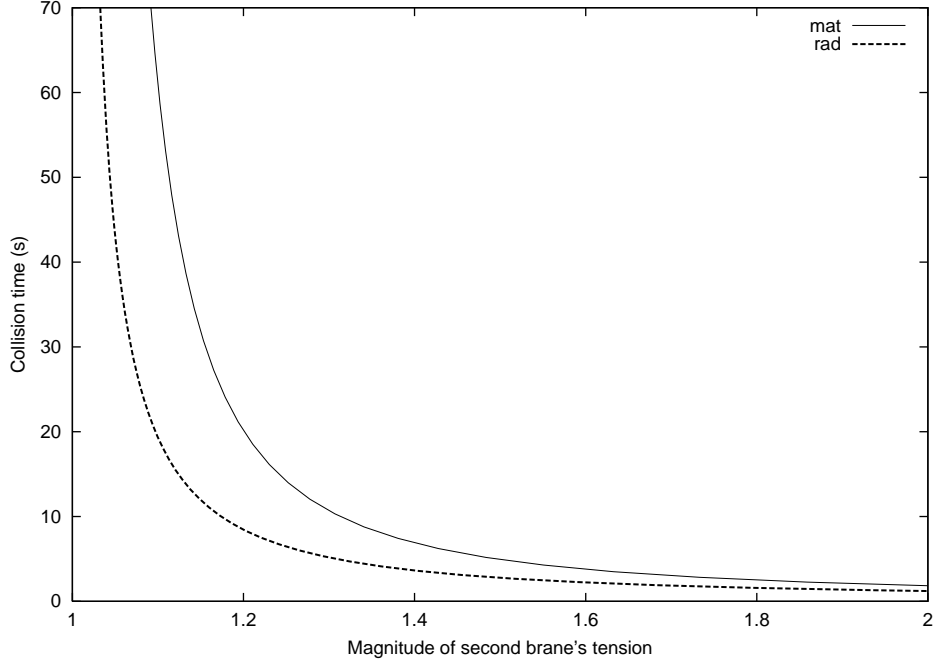


Figure 8: Time taken for branes to collide for varying η_2

model at late times, therefore the assumed form of η_2 and p_2 given by equation (5.26) is the only possible choice.

The junction conditions across the second brane give the energy density conservation equation:

$$\dot{\eta}_2 = -3H_2 (\eta_2 + p_2), \quad (5.27)$$

which in terms of η_{2p} takes the same form,

$$\dot{\eta}_{2p} = -3H_2 (\eta_{2p} + p_{2p}). \quad (5.28)$$

If the second brane has an equation of state of the form $p_{2p} = \omega_2 \eta_{2p}$ then we can use equation (5.28) to write,

$$\eta_{2p} = \left(\frac{\gamma_2}{a_2} \right)^{3(1+\omega_2)} = \left(\frac{\gamma_2}{a_2} \right)^{q_2}, \quad (5.29)$$

where we have defined $q_2 = 3(1 + \omega_2)$. A similar argument for the reference brane gives also,

$$\eta_{0p} = \left(\frac{\gamma_0}{a_0} \right)^{3(1+\omega_2)} = \left(\frac{\gamma_0}{a_0} \right)^{q_0}, \quad (5.30)$$

which was implicitly used in the previous two sections. γ_0 and γ_2 are simply constants relating the energy densities and scale factors on either brane.

One can now ask whether an equilibrium solution for the interbrane distance exists in this situation. Using the conditions for equilibrium given by equations (3.1)

and (3.3) we can write the equation for the possible equilibrium point z_e in the same form as equation (5.4),

$$z_e = \tanh^{-1} \frac{1}{(1 + \eta_{0p})} + \tanh^{-1} \frac{1}{(-1 - \eta_{2p})}. \quad (5.31)$$

Replacing η_{0p} and η_{2p} using equations (5.29) and (5.30), and rearranging gives,

$$z_e = \frac{1}{2}(q_0 - q_2) \ln a_0 - \frac{1}{2}q_2 \ln \left(\frac{a_2}{a_0} \right) + \frac{1}{2} \ln \left(\frac{\gamma_2^{q_2}}{\gamma_0^{q_0}} \right). \quad (5.32)$$

This shows that in general the required expression for z_e is time dependent (since both a_0 and a_2 depend on t) and that therefore no equilibrium position exists. If, however, we examine the late time behaviour of this equation then we can use the fact that $a_2/a_0 = \cosh z_e - \eta_0 \sinh z_e \approx \exp(-z_e)$ to first order⁴. Equation (5.32) can now be arranged to give,

$$z_e = \frac{1}{q_2 - 2} \left[(q_2 - q_0) \ln a_0 + \ln \left(\frac{\gamma_0^{q_0}}{\gamma_2^{q_2}} \right) \right] \quad (5.33)$$

This demonstrates that there will be an equilibrium solution at late times if and only if $q_0 = q_2$ i.e. if both branes have the same equation of state. Therefore if both branes are radiation dominated, $q_0 = q_2 = 4$ and $z_e = 2 \ln(\gamma_0/\gamma_2)$, however, if they are matter dominated then $q_0 = q_2 = 3$ and $z_e = 3 \ln(\gamma_0/\gamma_2)$.

We now investigate the stability of such solutions by examining equation (3.13), which in dimensionless variables and with $f_2 = 0$ looks like,

$$h_2 \delta z_{,s_2} + (1 - \bar{\eta}_2^2) \delta z = \delta \eta_2. \quad (5.34)$$

where we have defined $s_2 = \mu\tau$, the dimensionless time experienced by an observer on the second brane. Using the fact that we have set $\eta_2 = -1 - \eta_{2p} = -1 - (\gamma_2/a)^{q_2}$ gives,

$$\delta \eta_2 = q_2 \left(\frac{\gamma_2}{a} \right)^{q_2} \frac{a'}{a} \delta z = q_2 \bar{\eta}_{2p} (-1 - \bar{\eta}_{2p}) \delta z, \quad (5.35)$$

where we have used the fact that at equilibrium $a'/a = \bar{\eta}_2$. Combining equations (5.34) and (5.35) and expressing everything in terms of $\bar{\eta}_{2p}$ leads to,

$$\delta z_{,s_2} = - \left(\frac{\bar{\eta}_{2p}}{\bar{\eta}_{2p} + 2} \right)^{1/2} (q_2 - 2 + (q_2 - 1)\bar{\eta}_{2p}) \delta z. \quad (5.36)$$

This shows that the fluctuations around z_e will be stable as long as $q_2 > 2$. The explicit expression for η_{2p} in terms of the time experienced by an observer on the

⁴Technically we have had to assume both that $t \gg 1$ and that z_e is not significantly close to the coordinate singularity at $z_h = \tanh^{-1}(1/\eta_0)$.

second brane s_2 can be found from the Friedmann equation (2.17) combined with equation (5.29) and takes the same form as equation (5.23),

$$\eta_{2p}(s_2) = \frac{1}{\frac{1}{2}q_2^2 s_2^2 + q_2 s_2}. \quad (5.37)$$

Inserting this into the fluctuation equation allows us to solve for $\delta z(s_2)$ giving,

$$\delta z(s_2) = \frac{A(q_2 s_2 + 1)}{[q_2 s_2 (q_2 s_2 + 2)]^{(q_2 - 1)/q_2}}, \quad (5.38)$$

where A is an integration constant. Therefore at late times when $s_2 \gg 1$, the fluctuations will behave like $\delta z \sim (q_2 s_2)^{(2/q_2 - 1)}$ giving $\delta z_r \sim (q_2 s_2)^{-1/2}$ for radiation and $\delta z_m \sim (q_2 s_2)^{-1/3}$ for matter demonstrating that the interbrane distance will tend to z_e at late times. Incidentally, the early time behaviour is $\delta z_r \sim (q_2 s_2)^{-3/4}$ and $\delta z_m \sim (q_2 s_2)^{-2/3}$, and although we have already shown that the equilibrium position z_e is not constant at this time, provided it moves only slowly relative to the evolution of the fluctuations then these early time results will still be of significance.

Using equations (5.20), (5.21), (5.26), and (5.29) the radion equation (2.21) was solved numerically for four different cases, with both branes assumed to be in a state of either radiation or matter dominance and the results are shown in figures 9, 10, 11 and 12. In each case the six initial radion positions were distributed evenly between 0 and z_h , and the integration was started at $s = 0.1$. Figure 9 shows the evolution of the radion z against the dimensionless time variable s when both branes are radiation dominated in that $q_0 = q_2 = 4$. The proportionality constants have been chosen to be $\gamma_0 = 1.04$ and $\gamma_2 = 1$ and therefore the late time equilibrium position z_e given by equation (5.33) is $z_e \simeq 0.0784$ which is represented by the dotted line. At late time the stable behaviour predicted above is evident as all trajectories tend slowly towards z_e . The expected early time behaviour is also confirmed as initially the trajectories rapidly converge, however some of them possess stationary points which implies a time dependent solution for z_e . In fact, the minimum of a trajectory corresponds to where the time dependent solution for z_e actually crosses the trajectory. Figure 10 shows the radion's behaviour when both branes are matter dominated. Here the constants are $\gamma_0 = 1.02$ and $\gamma_2 = 1$ and therefore $z_e \simeq 0.0594$. A stable behaviour is again observed as expected from the above analysis, and the similarities with figure 9 are substantial. Note how the rate at which trajectories converge at late times is faster for radiation than for matter dominated branes in agreement with $\delta z_r \sim s^{-1/2}$ and $\delta z_m \sim s^{-1/3}$ as found above.

The interesting case in which the branes possess different equations of state is presented in figures 11 and 12. Figure 11 shows the evolution of the radion when the reference brane is radiation dominated and the second brane matter dominated. Initially, the trajectories rapidly converge in a similar manner to figures 9 and 10, however soon the disparity in the expansion rates of the two branes takes effect and

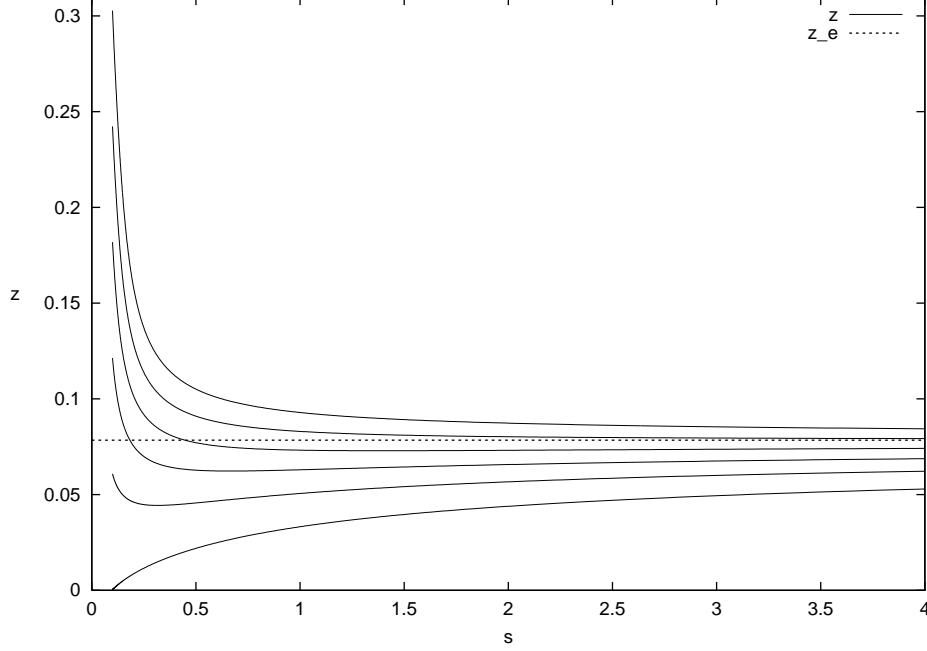


Figure 9: The evolution of the interbrane distance z when both branes are radiation dominated. The dotted line is the late time value of the equilibrium position z_e . Note the way some of the trajectories reach a minimum before approaching their limit: this is due to the early time dependence of z_e

inevitably the interbrane distance reaches zero, corresponding to the branes colliding. Figure 12 shows the opposite case, whereby the reference brane is matter dominated, and the second brane is radiation dominated. Again initially the trajectories rapidly converge, but since the reference brane is effectively moving faster through the bulk than the second brane, the interbrane distance increases indefinitely.

5.5 Phase Transitions on the Reference Brane

In this section we investigate the cosmological behaviour of the interbrane distance when phase transitions occur on the reference brane. It must be noted that here, unlike the previous section we keep the equation of state on the second brane to be $p_2 = -\eta_2$ - that of a constant brane tension.

Here we assume that the reference brane goes through three stages: accelerating, radiation dominated and matter dominated. In each stage we ignore all subdominant terms in the brane's energy density so that $\omega_0 = -1, 0, 1/3$ in each of the corresponding phases, leading to,

$$\eta_{0p} = \begin{cases} \lambda & s < s_1 \\ \lambda \frac{a_0(s_1)^4}{a_0(s)^4} & s_1 < s < s_2 \\ \lambda \frac{a_0(s_1)^4}{a_0(s_2)a_0(s)^3} & s_2 < s \end{cases}, \quad (5.39)$$

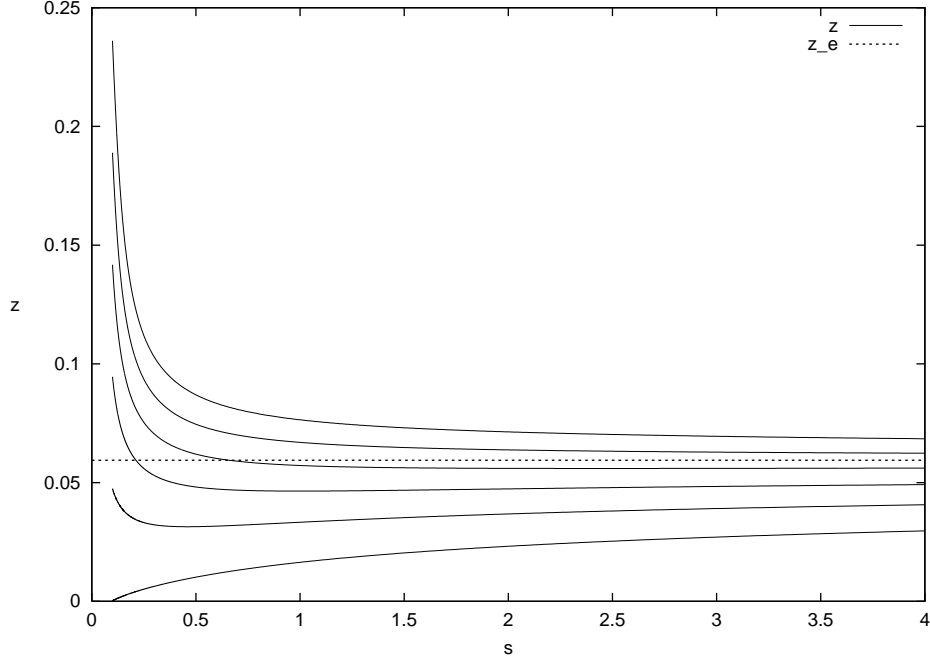


Figure 10: Matter on both branes

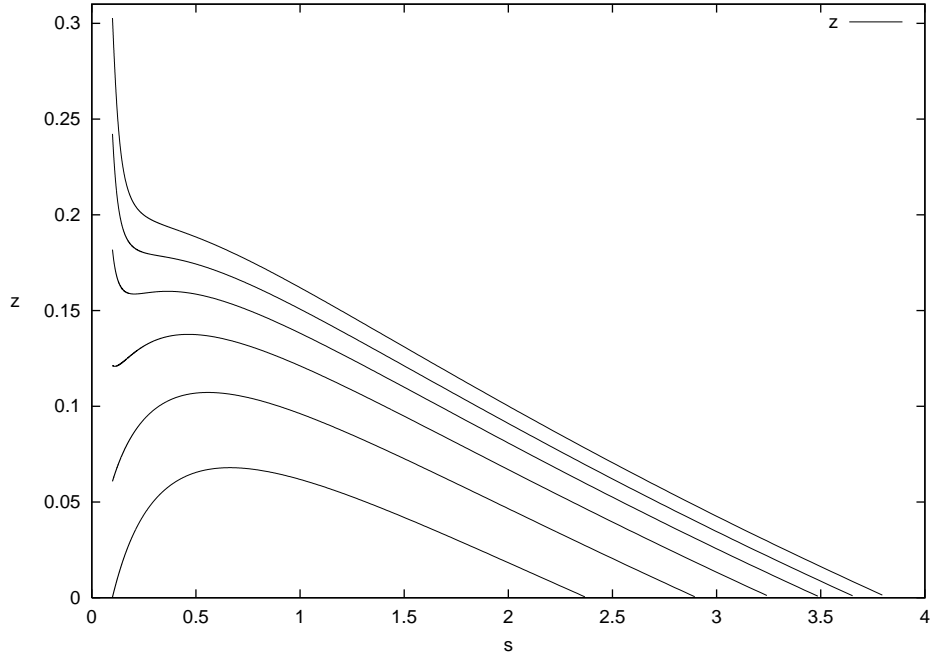


Figure 11: Radiation on the reference brane, matter on the second brane

where λ is a constant and s_1 and s_2 are the times at which the transitions occur. Note that here we assume that the transitions are instantaneous. Using this expression for η_{0p} and equations (5.17), (5.19), (5.20) and (5.21) the radion equation (2.21) was solved for two cases, each with several initial radion positions distributed evenly

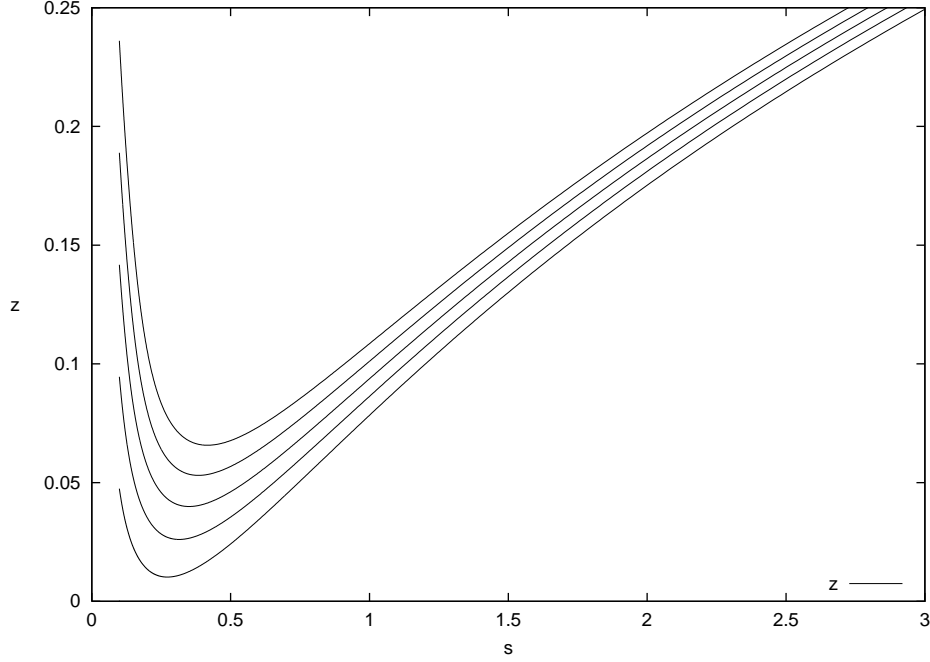


Figure 12: Matter on the reference brane, radiation on the second brane

between $z = 0$ and $z = 0.8z_h$, the results of which are shown in figures 13 and 14. Both cases had $\eta_2 = -1.2$ and initially $\eta_0 = 1.1$, and only differed by the choice of transition times s_1 and s_2 . The first graph, given in figure 13 has $s_1 = 4$, $s_2 \simeq 13$ and shows how during the first phase some of the radion trajectories quickly collide with the reference brane, and the rest tend toward z_h as would be expected from the analysis of constant tension branes in section 5.1. When the radiation phase occurs however, the surviving solutions quickly move away from the reference brane (in these brane based coordinates), before eventually returning to $z = 0$. The existence of the matter phase delays this return further. In the second case given by figure 14 the initial phase is twice as long as in the previous situation with $s_1 = 8$. The radiation-matter transition time has been chosen so that matter domination occurs at the same energy density as in the first case giving $s_2 \simeq 17$. Doubling s_1 has an interesting effect - the time taken for the inevitable collision between the branes is massively delayed, in fact it has been increased roughly by a factor of ten. This shows that a relatively short period of stronger inflation on the reference brane than on the second brane will delay the return of the second brane indefinitely. The fact that in these situations investigated here, the interbrane distance eventually reaches zero is of course dependent on the value of η_2 and the limiting value at late time of $\eta_0 = 1 + \eta_{0p}$. As expected, the behaviour described here and shown in the two figures is again in agreement with the conceptual arguments in terms of a bulk observer developed in the previous section.

We now go on to examine the interesting case of two positive tension non- Z_2

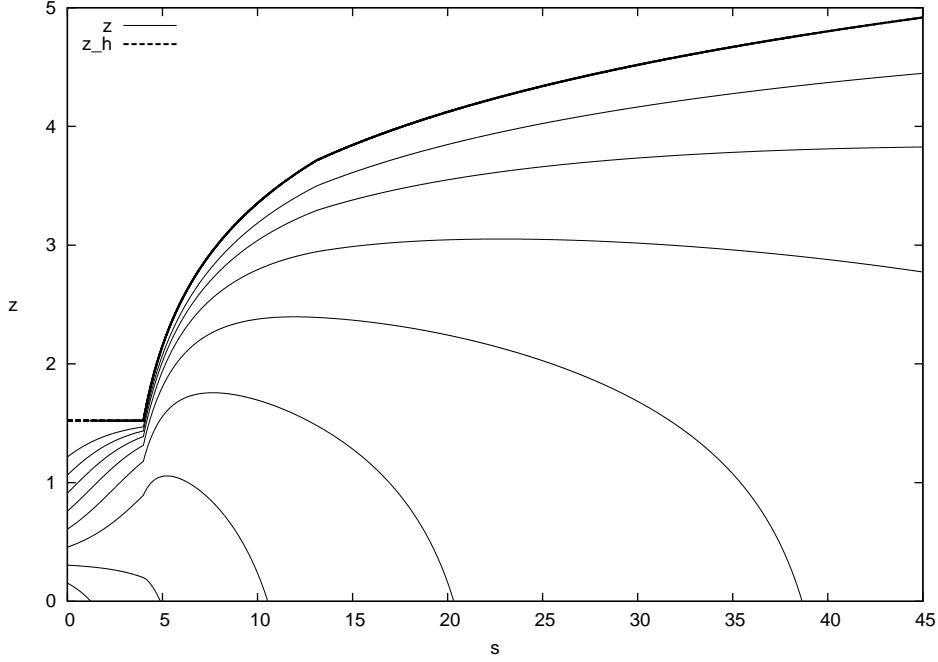


Figure 13: Phase transitions: initially the reference brane has a total constant energy density $\eta_0 = 1.1$, with $\eta_2 = -1.2$ on the second brane. At $t = 4$ and $t = 13$ the reference brane undergoes phase transitions into a radiation dominated universe and then a matter dominated universe respectively.

symmetric branes in a semi-infinite extra dimension.

5.6 Non- Z_2 Symmetry and Weyl Tensor Effects

As discussed in section 5.4, it is interesting to examine models where both branes possess cosmologically evolving energy densities however, often this requires a second brane tension η_2 , such that $\eta_2 \rightarrow -\infty$ as $t \rightarrow 0$. This could obviously lead to phenomenological difficulties, hence in this section we turn to an interesting model in which the reference brane is Z_2 symmetric, while the second brane is non- Z_2 symmetric allowing both branes to be of positive tension in a semi-infinite fifth dimension. A model of this kind was previously proposed by [58] as an attempt to solve the hierarchy problem using only positive tension branes. Several multi-brane extensions of this model have been studied [70–72] albeit not for cosmological cases.

Setting $\mathcal{C}_0 = \mathcal{C}_1 = k = 0$ but assuming $\mathcal{C}_2 \neq 0$, means that $h_0 = \eta_0^2 - 1$ and that the Friedmann equation for the second brane is given by,

$$h_2 = \eta_2^2 - 1 + \frac{\mathcal{C}_2}{2\mu^2 a_2^4} + \frac{\mathcal{C}_2^2}{16\mu^4 \eta_2^2 a_2^8}, \quad (5.40)$$

which can be rewritten using $f_2 = \mathcal{C}_2/4\mu$ as,

$$h_2 = \left(\eta_2 + \frac{f_2}{\mu \eta_2 a_2^4} \right)^2 - 1. \quad (5.41)$$

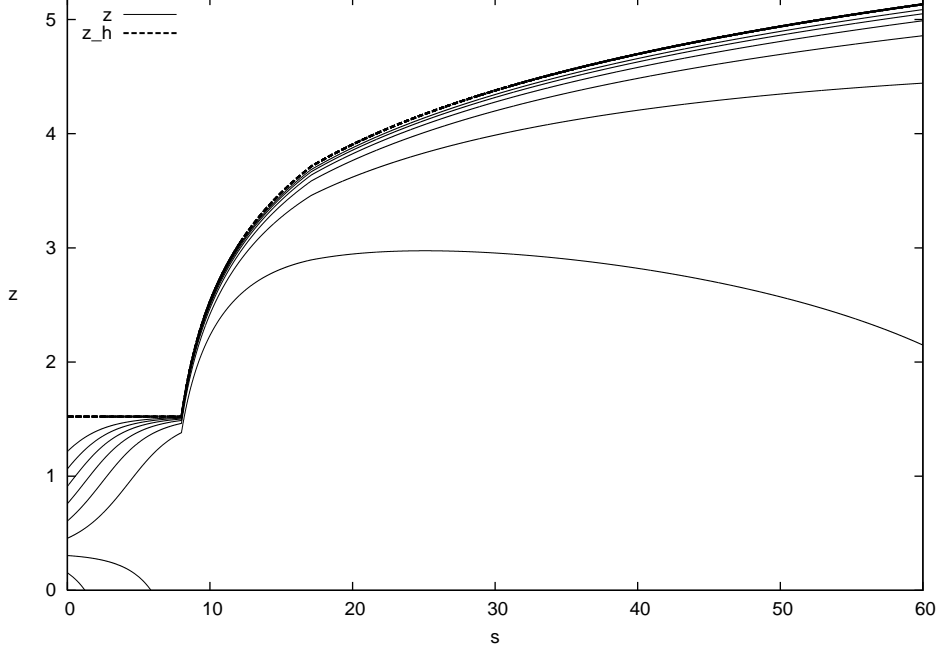


Figure 14: The initial phase is twice as long as fig 13 greatly delaying the return of the second brane.

Referring to equations (2.16) and (2.20) it can be seen that a solution of the two brane system with both branes of positive tension does exist in this situation, provided that,

$$\frac{-f_2}{\mu\eta_2 a_2^4} > \eta_2. \quad (5.42)$$

This implies both that $f_2 < 0$ and that the second brane must be dominated by its lack of Z_2 symmetry, a fact that prevents standard cosmology being realised on this brane. We studied the behaviour of the interbrane distance for two different equations of state on the second brane. The first was where the second brane possessed a constant brane tension $\eta_2 = 1.1$ and the results are shown in figure 15. The reference brane had a constant tension of $\eta_0 = 1.01$ and the non- Z_2 symmetric parameter was set to $f_2 = -1.0/\mu^3$ (in order to keep the radion equation independent of μ) and the six initial radion positions were distributed evenly between $z = 0.5$ and $z = 2.5$. Inspection of equation (5.41) shows that in this situation the second brane will initially expand until a maximum value of $a_2(\tau)$ is reached which satisfies,

$$a_{2max}(\tau) = \left(\frac{-f_2}{\mu\eta_2(1+\eta_2)} \right)^{1/4}. \quad (5.43)$$

Once this is reached $h_2 = 0$, and subsequently the second brane begins to contract. This behaviour helps to explain the trajectories shown in figure 15: the expanding second brane begins to catch up with the reference brane but once $a_2 \simeq a_{2max}$, the second brane's motion slows and it quickly falls behind and freezes out at $z_h \simeq 2.65$.

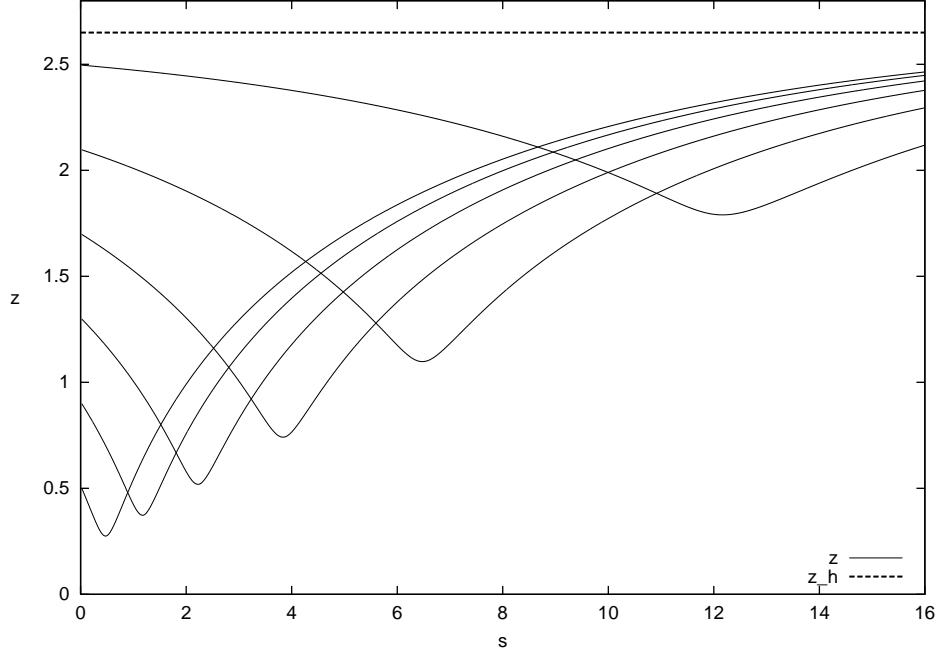


Figure 15: Radion trajectories for the two positive tension branes case where $\mu^3 f_2 = -1$, $\eta_0 = 1.01$ and $\eta_2 = 1.1$

In the second $f_2 \neq 0$ case investigated, we assumed $\omega_2 = 1/3$ and hence that the (positive) energy density η_2 was radiation dominated such that $\eta_2 = \gamma_r/a^4$, where γ_r is some constant. For each of the six trajectories, we set $\gamma_r = 5$ and chose a value of f_2 such that $\eta_2 + f/\mu\eta_2 a^4 = -1.1$ in order to ensure that the different trajectories all start with the same value of $h_2^2 = 0.21$. The reference brane was assumed to have $\eta_0 = 2.0$ and the results are shown in figure 16. In order to have two positive tension branes the inequality (5.42) was maintained and one can therefore see from equation (5.41) that initially h_2 is smaller than h_0 and the interbrane distance will increase, however the η_2 term in equation (5.41) soon becomes negligible and the second brane is then entirely dominated by the constant non- Z_2 symmetry such that $h_2 \simeq f_2^2/\mu^2\gamma_r^2 - 1$. Therefore at later times the situation is analogous to that of constant tension branes analysed in section 5.1: an unstable equilibrium position exists as can be seen in figure 16 and this agrees with the perturbative analysis of this situation presented in section 3.2.

Obviously there are many other two brane situations that could be investigated including both branes having no Z_2 -symmetry and both branes with non-zero Weyl tensor components, however, we leave this to a future investigation.

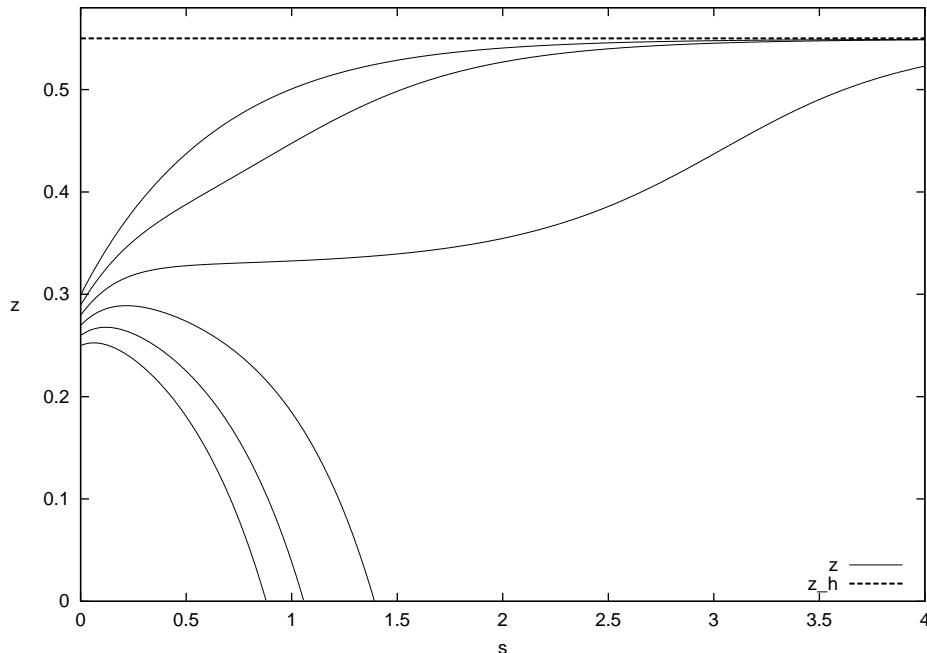


Figure 16: Radion trajectories where the reference brane possesses a constant brane tension and the second brane is radiation dominated and non- Z_2 symmetric.

6. Discussion

In this paper we have greatly extended the work done by Binetrui et al [51] on the cosmological radion of the two brane Randall Sundrum model. We started by using an elegant method to derive the non-linear equations of motion for the interbrane distance \mathcal{R} , and hence found a more general version of the radion equation, while sidestepping a lot of the algebra required in [51]. The $\ddot{\mathcal{R}}$ equation was derived and this allowed us to identify the general conditions for an equilibrium radion position. The equations of motion were then linearised in order to investigate the nature of the equilibrium positions and in the majority of cases these positions were found to be unstable, confirming and generalising previous work [47–49]. The radion equations were found to be extremely non-linear and it was therefore necessary to numerically integrate them in order to examine the radion’s behaviour far from the equilibrium positions. It should be noted that in some cases no equilibrium exists making a numerical analysis even more vital.

The numerical solutions confirmed that for two constant tension dS branes there sometimes existed an unstable equilibrium point, but for two AdS branes that point, if it exists would be stable. The cases where the reference brane possessed a more realistic equation of state, with standard cosmology taking over at late times were also examined, and for a constant second brane tension $\eta_2 > 1$ it was found that the branes inevitably collide. The time taken for this collision to occur was calculated and shown to depend upon among other things the initial radion position and the nature

of the matter on the reference brane, for example, radiation domination implies the branes collide sooner than if there was matter domination. It was demonstrated how if both branes possess a time dependent energy density then a stable equilibrium solution exists provided the branes possess the same equation of state. The effects of changes of state were also considered, and it was shown how a relatively short burst of inflation could greatly delay the return of the second brane. Two non- Z_2 symmetric scenarios involving two positive tension branes were investigated: one led to all trajectories freezing out at z_h , while the other at late times was analogous to the constant tension dS case. It should be noted that these scenarios exist in a semi-infinite space $y > 0$, and assuming $f_2 < 0$ implies that the Schwarzschild mass to the right of the second brane $\mathcal{C}_2 < 0$ which may lead to problems as the second brane will not be shielded from the naked singularity appearing in the bulk. This could be avoided by having for example $\mathcal{C}_0 = \mathcal{C}_1 > \mathcal{C}_2 > 0$ i.e. having all the Schwarzschild masses greater than zero.

It was discussed how, by considering the perspective of a bulk observer in a static bulk background, one can develop a conceptual understanding of many of the features of the radion displayed in this paper. Viewing the behaviour of the interbrane distance as a competition between the expansion rates of the branes, one can then analyse the branes' Friedmann equations to determine qualitatively the radion's behaviour. One may ask whether it would have been better to carry out this analysis in the bulk based coordinate system as opposed to a brane based one. However, we intend to extend this radion analysis to systems containing more complicated bulk matter than just a cosmological constant, namely scalar and dilaton fields. It is then not possible to use a time independent bulk based coordinate system since Birkhoff's theorem (the generalisation of Gauss' theorem) can no longer be applied. Given the elegance and ease of deriving the radion equation in the brane based system, the need to follow and extend the work done by [51], and the inevitable numerical calculation whatever coordinate system was used, it was felt that the brane based system was the most appropriate for our purposes.

Finally, it should be mentioned how special a case the cosmological radion is. Due to the amount of symmetry assumed, the trajectories of the branes are entirely determined by the knowledge of their Friedmann equations and hence their expansion rate, a fact which allowed us to efficiently derive the radion equation, and was not realised by [51] who had to use a more cumbersome method. In essence the branes do not 'feel' each other, a fact that is altered if one considers other fields in the bulk, or general perturbations to the cosmological background.

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